# Bandits on Graphs

*Exploiting smoothness and side observations*

**Michal Valko** (SequeL INRIA)

*joint work with*
**Shipra Agrawal** (MSR India)
**Tomáš Kocák** (SequeL INRIA)
**Branislav Kveton** (Technicolor $\rightarrow$ Adobe)
**Rémi Munos** (SequeL INRIA/Google Deepmind)
**Gergely Neu** (SequeL INRIA)

From the Creators of
MONSTERS, INC.
In Cinemas Soon
FROZEN
Harry Potter
THOR
DJANGO
LORD OF THE RINGS
SIN CITY
MUMMY
NICOLAS CAGE MEG RYAN
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NICHOLSON
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the BIG BANG theory
CBS
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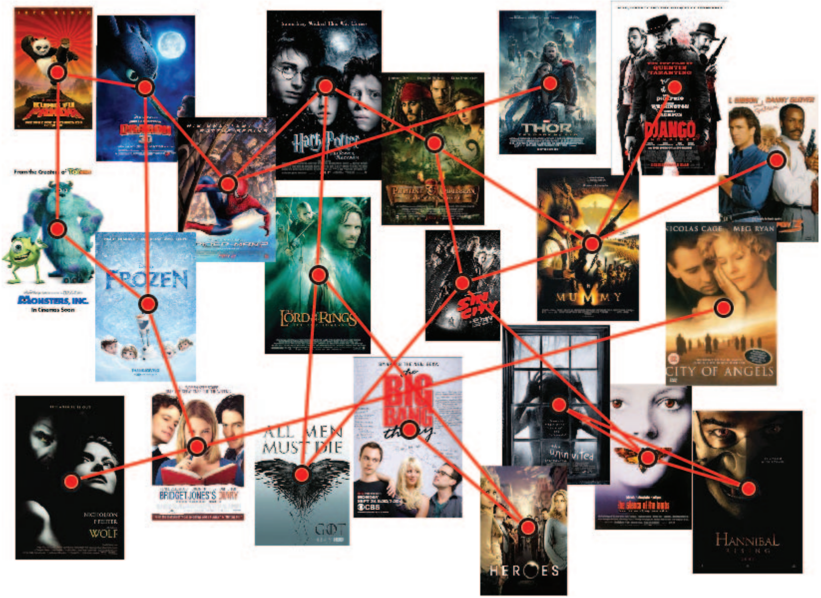
HANNIBAL



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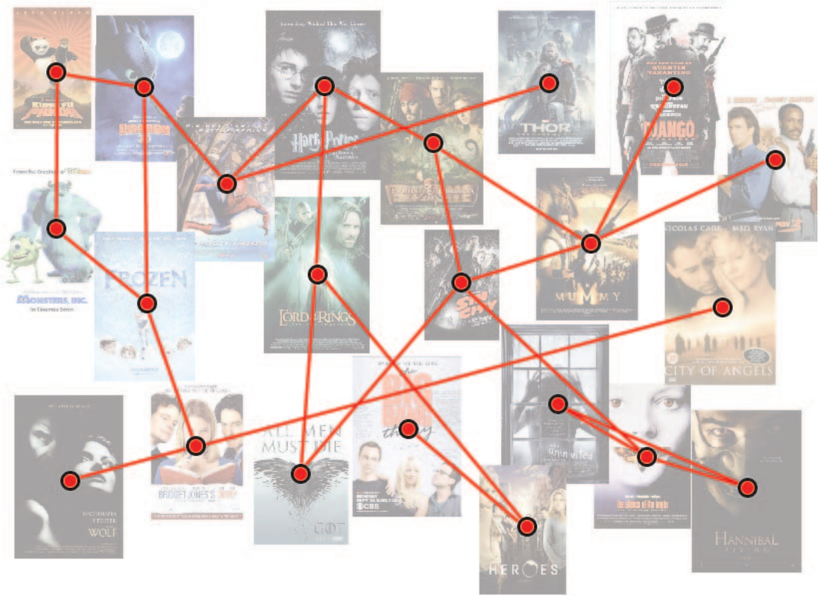

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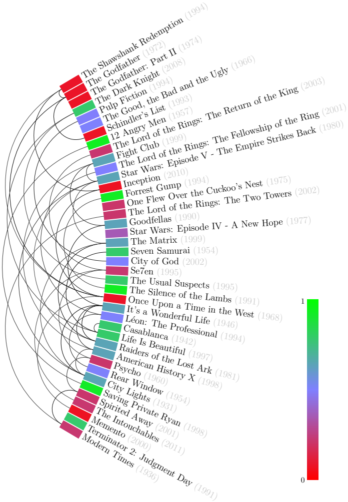
**Movie recommendation:** (in each time step)

- Recommend movies to a **single user**.



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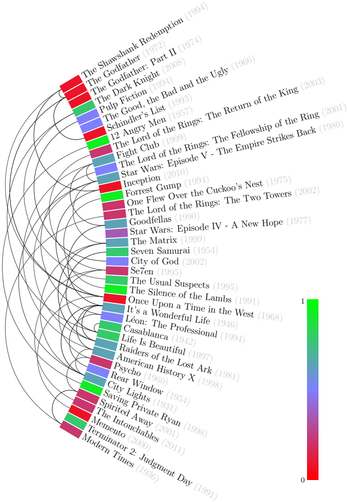

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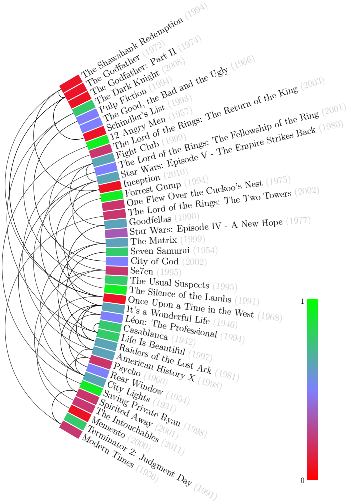

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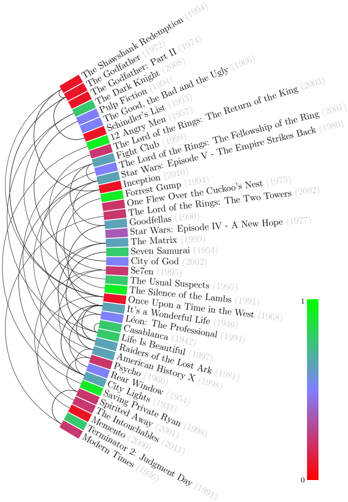
**Assumptions:**

- Unknown reward function $f : V(G) \to \mathbb{R}$.
- Function $f$ is **smooth** on a graph.
- Neighboring movies $\Rightarrow$ similar preferences.
- Similar preferences $\not\Rightarrow$ neighboring movies.



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**Smoothness of the function:**

$$S_G(f) = \frac{1}{2} \sum_{i,j \leq N} w_{i,j}(f_i - f_j)^2$$

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**Examples:**

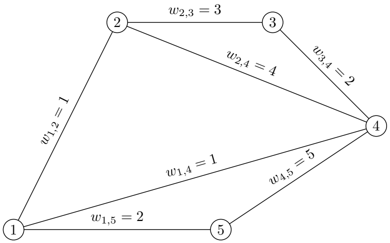
- **Complete graph:** Only constant function has smoothness 0.
- **Edgeless graph:** Every function has smoothness 0.
- **Constant function:** Smoothness 0 for every graph.

# Graph Laplacian

- $\mathcal{W}$: $N \times N$ matrix of the edge weights $w_{i,j}$.
- $\mathcal{D}$: Diagonal matrix with the entries $d_i = \sum_j w_{i,j}$.
- $\mathcal{L} = \mathcal{D} - \mathcal{W}$: Graph Laplacian.
  - Positive semidefinite matrix.
  - Diagonally dominant matrix.

**Example:**

$$\mathcal{L} = \begin{pmatrix} 4 & -1 & 0 & -1 & -2 \\ -1 & 8 & -3 & -4 & 0 \\ 0 & -3 & 5 & -2 & 0 \\ -1 & -4 & -2 & 12 & -5 \\ -2 & 0 & 0 & -5 & 7 \end{pmatrix}$$



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$$S_G(f) = \boldsymbol{f}^\mathsf{T}\mathcal{L}\boldsymbol{f} = \boldsymbol{f}^\mathsf{T}\mathbf{Q}\boldsymbol{\Lambda}\mathbf{Q}^\mathsf{T}\boldsymbol{f} = \boldsymbol{\alpha}^{*\mathsf{T}}\boldsymbol{\Lambda}\boldsymbol{\alpha}^* = \|\boldsymbol{\alpha}^*\|_{\boldsymbol{\Lambda}}^2 = \sum_{i=1}^{N} \lambda_i (\alpha_i^*)^2$$

**Smoothness and regularization:** Small value of

**(a)** $S_G(f)$   **(b)** $\boldsymbol{\Lambda}$ norm of $\boldsymbol{\alpha}^*$   **(c)** $\alpha_i^*$ for large $\lambda_i$

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- In each time $t$ step choose a node $\pi(t)$.
- the $\pi(t)$-th row $\mathbf{x}_{\pi(t)}$ of the matrix $\mathbf{Q}$ corresponds to the arm $\pi(t)$.
- Obtain noisy reward $r_t = \mathbf{x}_{\pi(t)}^{\mathsf{T}} \boldsymbol{\alpha}^* + \varepsilon_t$. Note: $\mathbf{x}_{\pi(t)}^{\mathsf{T}} \boldsymbol{\alpha}^* = f_{\pi(t)}$
  - $\varepsilon_t$ is $R$-sub-Gaussian noise. $\forall \xi \in \mathbb{R}, \mathbb{E}[e^{\xi \varepsilon_t}] \leq \exp\left(\xi^2 R^2 / 2\right)$
- Minimize cumulative regret

$$R_T = T \max_a \left(\mathbf{x}_a^{\mathsf{T}} \boldsymbol{\alpha}^*\right) - \sum_{t=1}^{T} \mathbf{x}_{\pi(t)}^{\mathsf{T}} \boldsymbol{\alpha}^*.$$

- Can't we just use *linear bandits*?

# Solutions

- **Linear bandit algorithms** (Existing solutions)
  - **LinUCB** (Li et al., 2010)
    - Regret bound $\approx D\sqrt{T \ln T}$
  - **LinearTS** (Agrawal and Goyal, 2013)
    - Regret bound $\approx D\sqrt{T \ln N}$

**Note:** $D$ is ambient dimension, in our case $N$, length of $x_i$.
Number of actions, e.g., all possible movies $\rightarrow$ **HUGE!**

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  - **SpectralTS** (Kocák et al., AAAI 2014)
    - Regret bound $\approx d\sqrt{T \ln N}$
    - Operations per step: $D^2 + DN$

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# Effective dimension

- **Effective dimension:** Largest $d$ such that

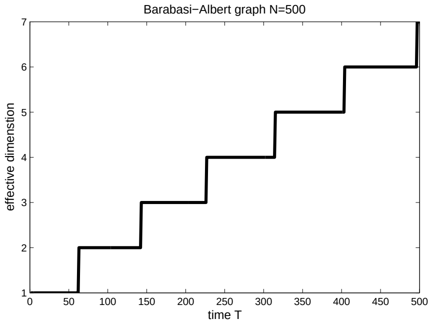
$$(d-1)\lambda_d \leq \frac{T}{\log(1 + T/\lambda)}.$$

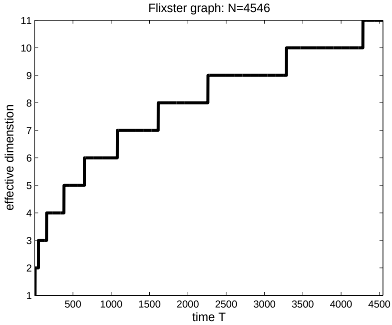
- Function of time horizon and graph properties
- $\lambda_i$ : $i$-th smallest eigenvalue of $\mathbf{\Lambda}$.
- $\lambda$: Regularization parameter of the algorithm.

**Properties:**

- $d$ is small when the coefficients $\lambda_i$ grow rapidly above time.
- $d$ is related to the number of "non-negligible" dimensions.
- Usually $d$ is much smaller than D in real world graphs.
- Can be computed beforehand.

# Effective dimension vs. Ambient dimension

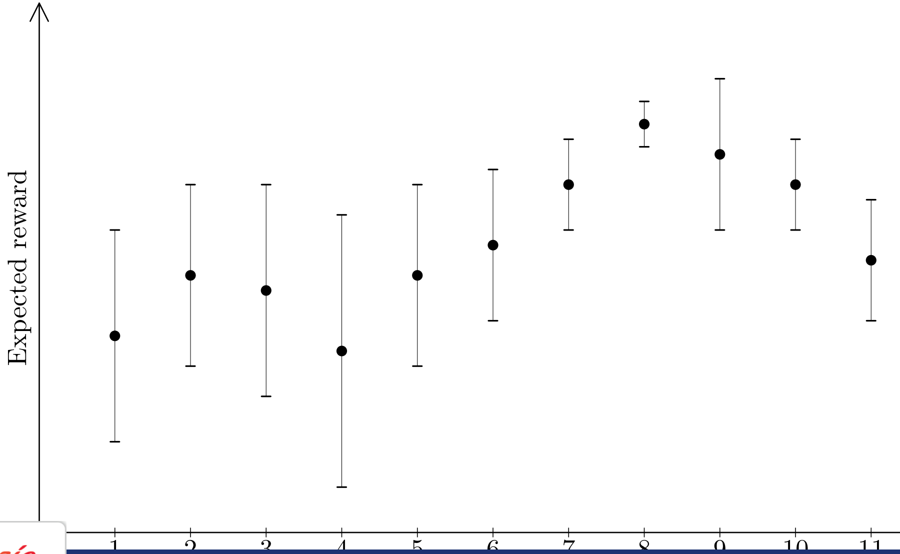


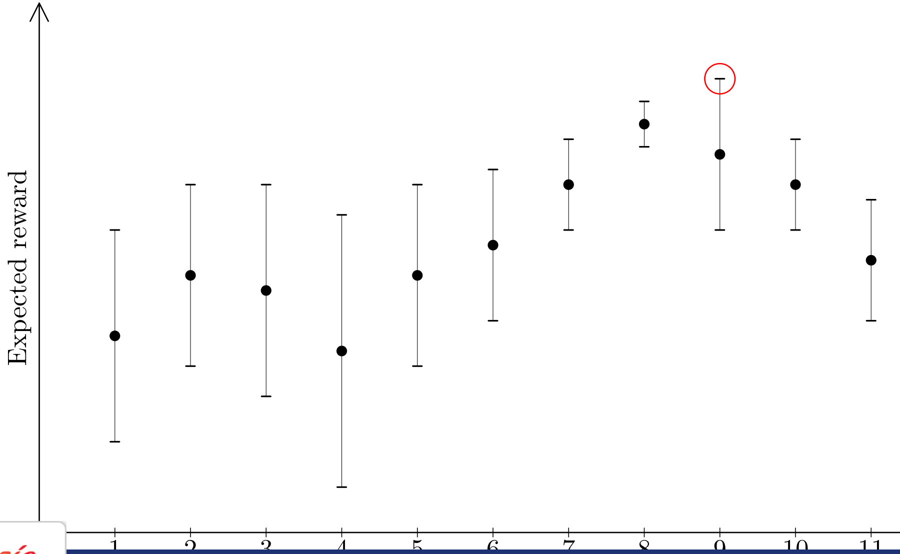

$$d \ll D$$

Note: In our setting $T < N = D$.

# UCB style algorithms: **Estimate**



# UCB style algorithms: **Sample**



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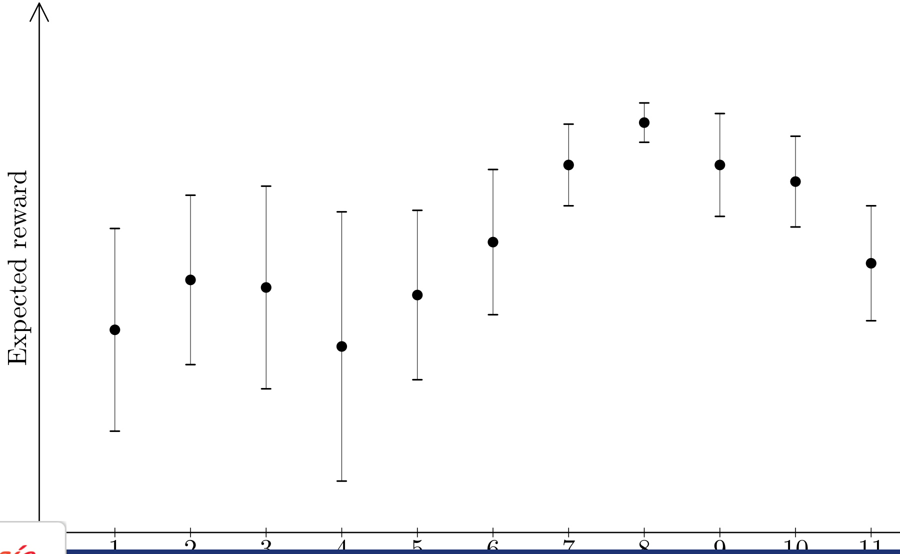

# SpectralUCB

**Input:**
  $N$, $T$, $\{\mathbf{\Lambda}_{\mathcal{L}}, \mathbf{Q}\}$, $\lambda$, $\delta$, $R$, $C$ $\mathcal{L}$
**Run:**
  $\mathbf{\Lambda} \leftarrow \mathbf{\Lambda}_{\mathcal{L}} + \lambda \mathbf{I}$
  $d \leftarrow \max\{d : (d-1)\lambda_d \leq T/\ln(1+T/\lambda)\}$
**for** $t = 1$ **to** $T$ **do**
  Update the basis coefficients $\hat{\boldsymbol{\alpha}}$:
  $\mathbf{X}_t \leftarrow [\mathbf{x}_{\pi(1)}, \ldots, \mathbf{x}_{\pi(t-1)}]^{\mathsf{T}}$
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  $c_t \leftarrow 2R\sqrt{d \ln(1 + t/\lambda) + 2\ln(1/\delta)} + C$
  $\pi(t) \leftarrow \arg\max_a \left( \mathbf{x}_a^{\mathsf{T}} \hat{\boldsymbol{\alpha}} + c_t \|\mathbf{x}_a\|_{\mathbf{V}_t^{-1}} \right)$
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$\mathbf{V}_t \leftarrow \mathbf{X}_t \mathbf{X}_t^\mathsf{T} + \mathbf{\Lambda}$
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$c_t \leftarrow 2R\sqrt{d \ln(1 + t/\lambda) + 2\ln(1/\delta)} + C$
$\pi(t) \leftarrow \arg\max_a \left( \mathbf{x}_a^\mathsf{T} \hat{\boldsymbol{\alpha}} + c_t \|\mathbf{x}_a\|_{\mathbf{V}_t^{-1}} \right)$
Observe the reward $r_t$
**end for**

# SpectralUCB

**Input:**
$N,\ T,\ \{\mathbf{\Lambda}_{\mathcal{L}}, \mathbf{Q}\},\ \lambda,\ \delta,\ R,\ C\ \mathcal{L}$
**Run:**
$\mathbf{\Lambda} \leftarrow \mathbf{\Lambda}_{\mathcal{L}} + \lambda \mathbf{I}$
$d \leftarrow \max\{d : (d-1)\lambda_d \leq T / \ln(1 + T/\lambda)\}$
**for** $t = 1$ **to** $T$ **do**
Update the basis coefficients $\hat{\boldsymbol{\alpha}}$:
$\mathbf{X}_t \leftarrow [\mathbf{x}_{\pi(1)}, \ldots, \mathbf{x}_{\pi(t-1)}]^\mathsf{T}$
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Observe the reward $r_t$
**end for**

# SpectralUCB regret bound

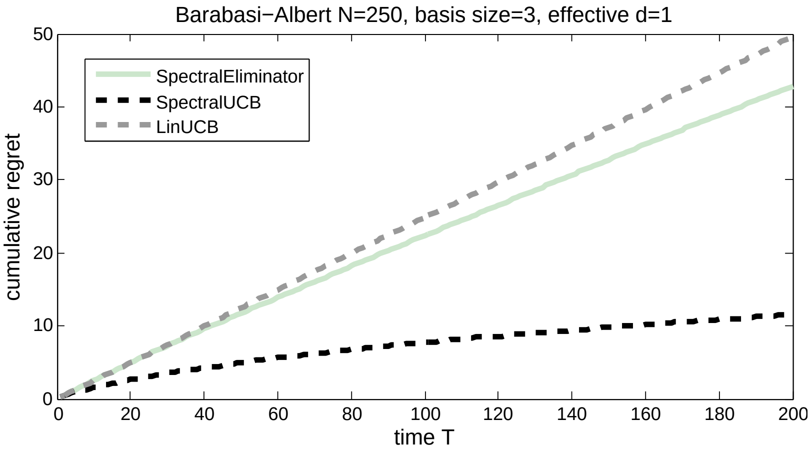
- $d$: Effective dimension.
- $\lambda$: Minimal eigenvalue of $\mathbf{\Lambda} = \mathbf{\Lambda}_{\mathcal{L}} + \lambda \mathbf{I}$.
- $C$: Smoothness upper bound, $\|\boldsymbol{\alpha}^*\|_{\mathbf{\Lambda}} \leq C$.
- $\mathbf{x}_i^\mathsf{T} \boldsymbol{\alpha}^* \in [-1, 1]$ for all $i$.

The **cumulative regret** $R_T$ of **SpectralUCB** is with probability $1 - \delta$ bounded as

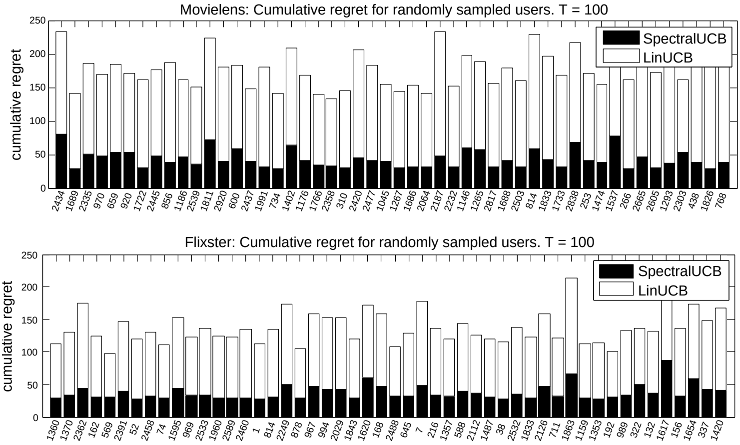
$$R_T \leq \left( 8R\sqrt{d \ln \frac{\lambda + T}{\lambda} + 2 \ln \frac{1}{\delta}} + 4C + 4 \right) \sqrt{dT \ln \frac{\lambda + T}{\lambda}}.$$

$$R_T \approx d\sqrt{T \ln T}$$

# Synthetic experiment

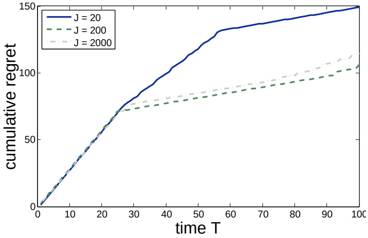


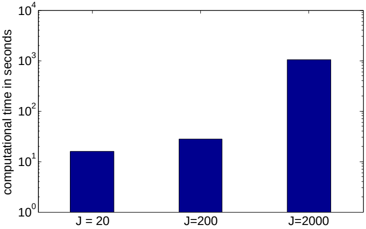
# Real world experiment



# Improving the running time: reduced eigenbasis

- **Reduced basis:** We only need first few eigenvectors.
- **Getting $J$ eigenvectors:** $\mathcal{O}(Jm \log m)$ time for $m$ edges
- Computationally less expensive, comparable performance.



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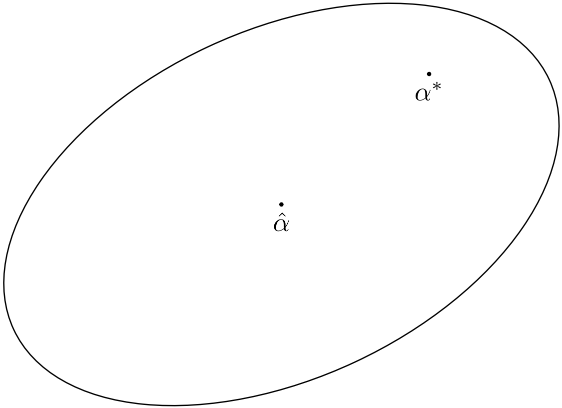
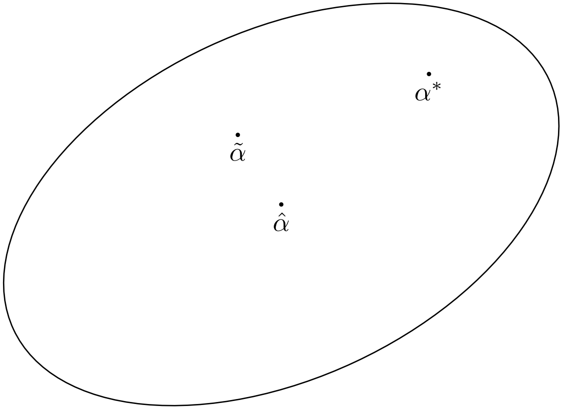
  - Compute posterior distribution according to reward received
- Only requires $D^2 + DN \rightarrow N^2$ per step update

# Thomson Sampling: **Estimate**



# Thomson Sampling: **Sample**



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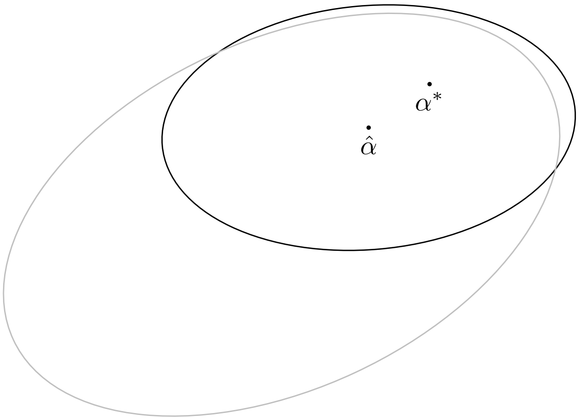

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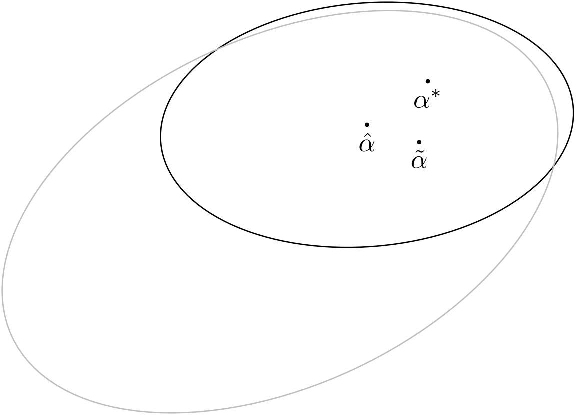
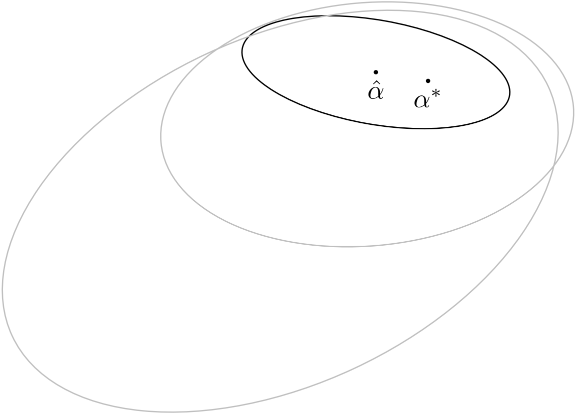


# Thomson Sampling: **Estimate** ...



# SpectralTS algorithm

**Input:**
    $N$, $T$, $\{\mathbf{\Lambda}_{\mathcal{L}}, \mathbf{Q}\}$, $\lambda$, $\delta$, $R$, $C$
**Initialization:**
    $v = R\sqrt{6d\log((\lambda + T)/\delta\lambda)} + C$
    $\hat{\boldsymbol{\alpha}} = 0_N$
    $\boldsymbol{f} = 0_N$
    $\mathbf{V} = \mathbf{\Lambda}_{\mathcal{L}} + \lambda\mathbf{I}_N$
**Run:**
**for** $t = 1$ **to** $T$ **do**
    Sample $\tilde{\boldsymbol{\alpha}} \sim \mathcal{N}(\hat{\boldsymbol{\alpha}}, v^2\mathbf{V}^{-1})$
    $\pi(t) \leftarrow \arg\max_a \mathbf{x}_a^{\mathsf{T}}\tilde{\boldsymbol{\alpha}}$
    Observe a noisy reward $r(t) = \mathbf{x}_{\pi(t)}^{\mathsf{T}}\boldsymbol{\alpha}^* + \varepsilon_t$
    $\boldsymbol{f} \leftarrow \boldsymbol{f} + \mathbf{x}_{\pi(t)}r(t)$
    Update $\mathbf{V} \leftarrow \mathbf{V} + \mathbf{x}_{\pi(t)}\mathbf{x}_{\pi(t)}^{\mathsf{T}}$
    Update $\hat{\boldsymbol{\alpha}} \leftarrow \mathbf{V}^{-1}\boldsymbol{f}$
**end for**

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# SpectralTS regret bound

- $d$: Effective dimension.
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- $\mathbf{x}_i^\mathsf{T}\boldsymbol{\alpha}^* \in [-1, 1]$ for all $i$.

The **cumulative regret** $R_T$ of **SpectralTS** is with probability $1 - \delta$ bounded as

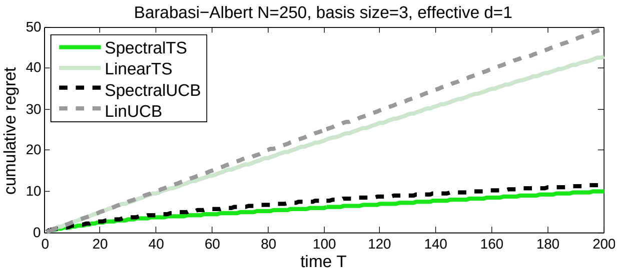
$$\mathcal{R}_T \leq \frac{11g}{p}\sqrt{\frac{4+4\lambda}{\lambda}dT\log\frac{\lambda+T}{\lambda}} + \frac{1}{T} + \frac{g}{p}\left(\frac{11}{\sqrt{\lambda}}+2\right)\sqrt{2T\log\frac{2}{\delta}},$$

where $p = 1/(4e\sqrt{\pi})$ and

$$g = \sqrt{4\log TN}\left(R\sqrt{6d\log\left(\frac{\lambda+T}{\delta\lambda}\right)} + C\right) + R\sqrt{2d\log\left(\frac{(\lambda+T)T^2}{\delta\lambda}\right)} + C.$$

$$R_T \approx d\sqrt{T\log N}$$

# Synthetic experiment



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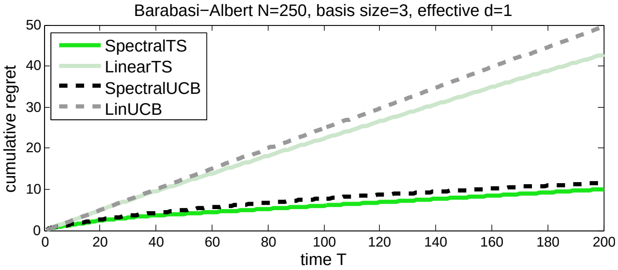
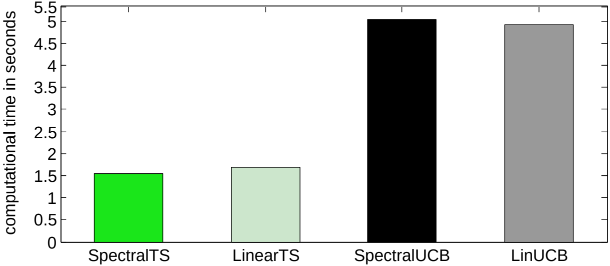
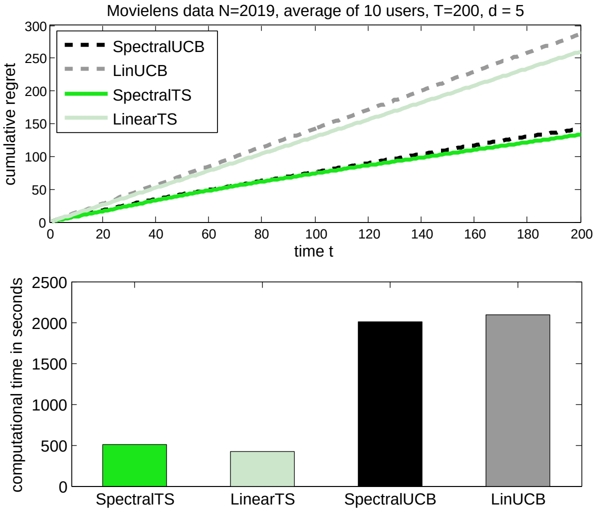


# Real world experiment

MovieLens dataset of 6k users who rated one million movies.



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## Example 1: undirected observations

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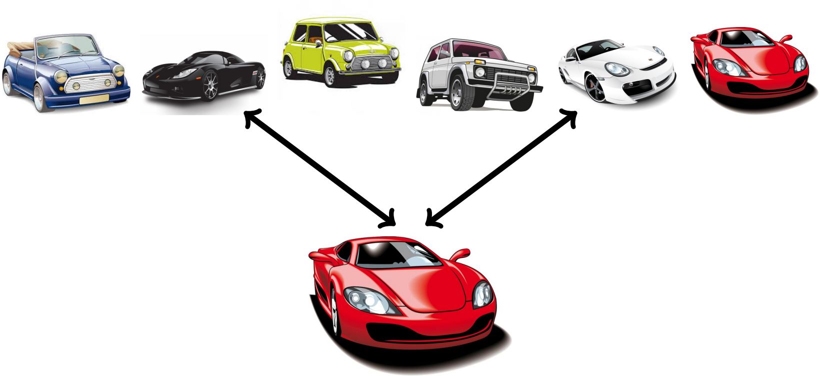
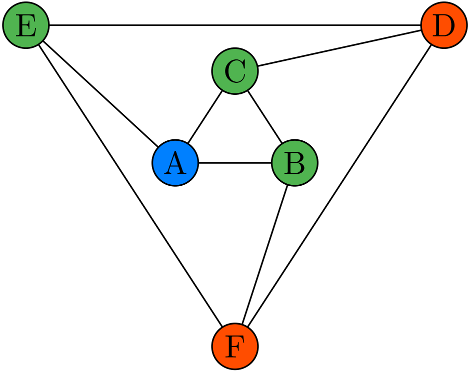


## Example 1: Graph Representation



# Example 2: Directed observation

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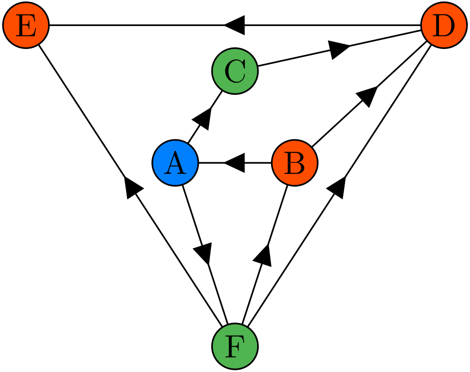

## Learning setting

**In each time step** $t = 1, \ldots, T$

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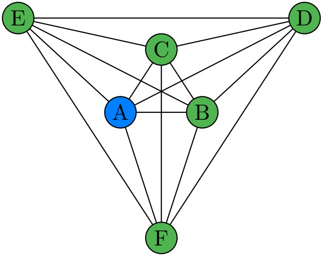
    - Graph: disclosed
- **Performance measure:** Total expected regret

$$R_T = \max_{i \in [N]} \mathbb{E}\left[\sum_{t=1}^{T} (\ell_{t,I_t} - \ell_{t,i})\right]$$

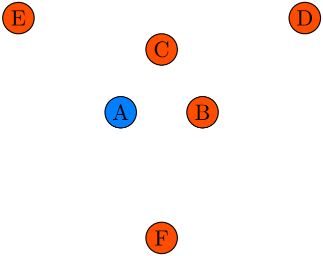
**Full Information setting**

- Pick an action (e.g. action A)
- Observe losses of all actions
- $R_T = \widetilde{\mathcal{O}}(\sqrt{T})$




**Bandit setting**

- Pick an action (e.g. action A)
- Observe loss of a chosen action
- $R_T = \widetilde{\mathcal{O}}(\sqrt{NT})$



**Side observation (Undirected case)**

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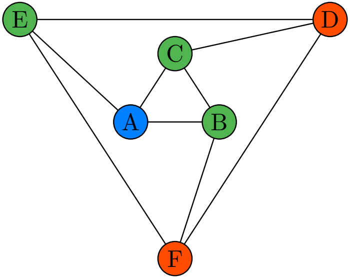

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**Mannor and Shamir** (ELP algorithm)

- Need to know graph
- Clique decomposition ($c$ cliques)
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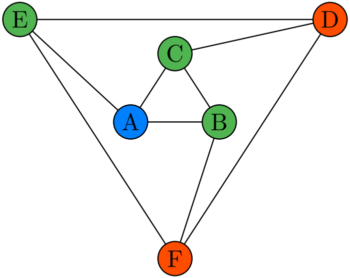


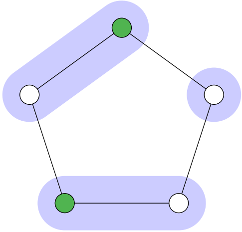

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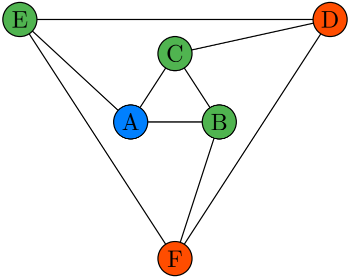
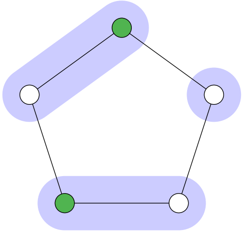
**Alon, Cesa-Bianchi, Gentile, Mansour**

- No need to know graph
- Independence set of $\alpha$ actions
- $R_T = \widetilde{\mathcal{O}}(\sqrt{\alpha T})$



## Side observation (Directed case)

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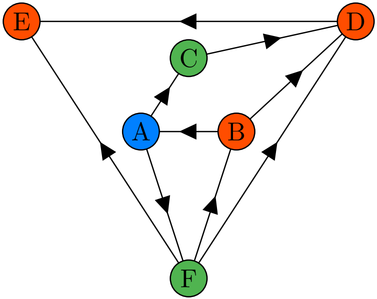

**Side observation (Directed case)**

- Pick an action (e.g. action A)
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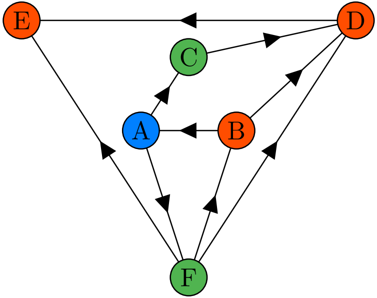
**Alon, Cesa-Bianchi, Gentile, Mansour**

- **Exp3-DOM**
- Need to know graph
- Need to find dominating set
- $R_T = \widetilde{\mathcal{O}}(\sqrt{\alpha T})$



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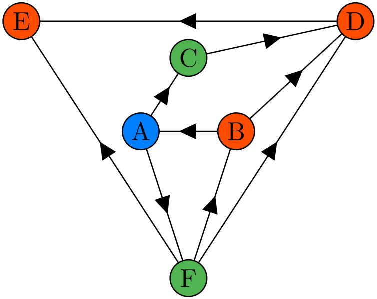
**Our solution: Exp3-IX**

- No need to know graph
- $R_T = \widetilde{\mathcal{O}}(\sqrt{\alpha T})$



# Exp3 algorithms in general

- **Compute weights** using loss estimates $\hat{\ell}_{t,i}$.

$$w_{t,i} = \exp\left(-\eta \sum_{s=1}^{t-1} \hat{\ell}_{s,i}\right)$$

- **Play action** $I_t$ such that

$$\mathbb{P}(I_t = i) = p_{t,i} = \frac{w_{t,i}}{W_t} = \frac{w_{t,i}}{\sum_{j=1}^{N} w_{t,j}}$$

- **Update loss estimates** (using observability graph)

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**How the algorithms approach to bias variance tradeoff?**

## Bias variance tradeoff approaches

- Approach of previous algorithms – Mixing
  - Bias sampling distribution $\mathbf{p}_t$ over actions
    - $\mathbf{p}_t' = (1-\gamma)\mathbf{p}_t + \gamma \mathbf{s}_t$ – mixed distribution
    - $\mathbf{s}_t$ – probability distribution which supports exploration
  - Loss estimates $\hat{\ell}_{t,i}$ are unbiased
- Approach of our algorithm – Implicit eXploration (**IX**)
  - Bias loss estimates $\hat{\ell}_{t,i}$
    - Biased loss estimates $\implies$ biased weights
    - Biased weights $\implies$ biased probability distribution
  - No need for mixing

## Mannor and Shamir - ELP algorithm

- $\mathbb{E}[\hat{\ell}_{t,i}] = \ell_{t,i}$ – unbiased loss estimates
- $p'_{t,i} = (1-\gamma)p_{t,i} + \gamma s_{t,i}$ – bias by mixing
- $\mathbf{s}_t = \{s_{t,1}, \ldots, s_{t,N}\}$ – probability distribution over the action set

$$\mathbf{s}_t = \arg\max_{\mathbf{s}_t} \left[ \min_{j \in [N]} \left( s_{t,j} + \sum_{k \in N_{t,j}} s_{t,k} \right) \right] = \arg\max_{\mathbf{s}_t} \left[ \min_{j \in [N]} q_{t,j} \right]$$

- $q_{t,j}$ – probability that loss of $j$ is observed according to $\mathbf{s}_t$

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- $q_{t,j}$ – probability that loss of $j$ is observed according to $\mathbf{s}_t$
- **Computation of $\mathbf{s}_t$**
  - Graph needs to be disclosed
  - Solving simple linear program
- Needs to know graph before playing an action
- Graphs can be only undirected

## Alon, Cesa-Bianchi, Gentile, Mansour - Exp3-DOM

- $\mathbb{E}[\hat{\ell}_{t,i}] = \ell_{t,i}$ – unbiased loss estimates
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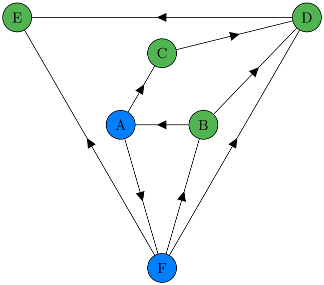
- $\mathbf{s}_t = \{s_{t,1}, \ldots, s_{t,N}\}$ – probability distribution over the action set

$$s_{t,i} = \begin{cases} \frac{1}{r} & \text{if } i \in R;\ |R| = r \\ 0 & \text{otherwise.} \end{cases}$$




- $R$ – dominating set of $r$ elements
- $\mathbf{s}_t$ – uniform distribution over $R$
- Needs to know graph beforehand
- Graphs can be directed

## Previous algorithms - loss estimates

$$\hat{\ell}_{t,i} = \begin{cases} \ell_{t,i}/o_{t,i} & \text{if } \ell_{t,i} \text{ is observed} \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathbb{E}[\hat{\ell}_{t,i}] = \frac{\ell_{t,i}}{o_{t,i}} o_{t,i} + 0(1 - o_{t,i}) = \ell_{t,i}$$

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## **Exp3-IX** - loss estimates

$$\hat{\ell}_{t,i} = \begin{cases} \ell_{t,i}/(o_{t,i} + \gamma) & \text{if } \ell_{t,i} \text{ is observed} \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathbb{E}[\hat{\ell}_{t,i}] = \frac{\ell_{t,i}}{o_{t,i} + \gamma} o_{t,i} + 0(1 - o_{t,i}) = \ell_{t,i} - \ell_{t,i} \frac{\gamma}{o_{t,i} + \gamma} \leq \ell_{t,i}$$

- No mixing!

# Analysis of Exp3 algorithms in general

- Evolution of $W_{t+1}/W_t$

$$\frac{1}{\eta}\log\frac{W_{t+1}}{W_t} = \frac{1}{\eta}\log\left(1-\eta\sum_{i=1}^{N}p_{t,i}\hat{\ell}_{t,i}+\frac{\eta^2}{2}\sum_{i=1}^{N}p_{t,i}(\hat{\ell}_{t,i})^2\right),$$

$$\sum_{i=1}^{N}p_{t,i}\hat{\ell}_{t,i} \leq \left[\frac{\log W_t}{\eta}-\frac{\log W_{t+1}}{\eta}\right]+\frac{\eta}{2}\sum_{i=1}^{N}p_{t,i}(\hat{\ell}_{t,i})^2$$

- Taking expectation and summing over time

$$\mathbb{E}\left[\sum_{t=1}^{T}\sum_{i=1}^{N}p_{t,i}\hat{\ell}_{t,i}\right]-\mathbb{E}\left[\sum_{t=1}^{T}\hat{\ell}_{t,k}\right] \leq \mathbb{E}\left[\frac{\log N}{\eta}\right]+\mathbb{E}\left[\frac{\eta}{2}\sum_{t=1}^{T}\sum_{i=1}^{N}p_{t,i}(\hat{\ell}_{t,i})^2\right]$$

# Regret bound of Exp3-IX

$$\underbrace{\mathbb{E}\left[\sum_{t=1}^{T}\sum_{i=1}^{N} p_{t,i}\hat{\ell}_{t,i}\right]}_{A} - \underbrace{\mathbb{E}\left[\sum_{t=1}^{T}\hat{\ell}_{t,k}\right]}_{B} \leq \mathbb{E}\left[\frac{\log N}{\eta}\right] + \underbrace{\mathbb{E}\left[\frac{\eta}{2}\sum_{t=1}^{T}\sum_{i=1}^{N} p_{t,i}(\hat{\ell}_{t,i})^2\right]}_{C}$$

**Lower bound of A** (using definition of loss estimates)

$$\mathbb{E}\left[\sum_{t=1}^{T}\sum_{i=1}^{N} p_{t,i}\hat{\ell}_{t,i}\right] \geq \mathbb{E}\left[\sum_{t=1}^{T}\sum_{i=1}^{N} p_{t,i}\ell_{t,i}\right] - \mathbb{E}\left[\gamma\sum_{t=1}^{T}\sum_{i=1}^{N}\frac{p_{t,i}}{o_{t,i}+\gamma}\right]$$

**Lower bound of B** (optimistic loss estimates: $\mathbb{E}[\hat{\ell}] < \mathbb{E}[\ell]$)

$$-\mathbb{E}\left[\sum_{t=1}^{T}\hat{\ell}_{t,k}\right] \geq -\mathbb{E}\left[\sum_{t=1}^{T}\ell_{t,k}\right]$$

**Upper bound of C** (using definition of loss estimates)

$$\mathbb{E}\left[\frac{\eta}{2}\sum_{t=1}^{T}\sum_{i=1}^{N} p_{t,i}(\hat{\ell}_{t,i})^2\right] \leq \mathbb{E}\left[\frac{\eta}{2}\sum_{t=1}^{T}\sum_{i=1}^{N}\frac{p_{t,i}}{o_{t,i}+\gamma}\right]$$

## Regret bound of Exp3-IX

$$R_T \leq \frac{\log N}{\eta} + \left(\frac{\eta}{2} + \gamma\right) \sum_{t=1}^{T} \mathbb{E}\left[\sum_{i=1}^{N} \frac{p_{t,i}}{o_{t,i} + \gamma}\right]$$

$$R_T \approx \mathcal{O}\left(\sqrt{\log N \sum_{t=1}^{T} \mathbb{E}\left[\sum_{i=1}^{N} \frac{p_{t,i}}{o_{t,i} + \gamma}\right]}\right)$$

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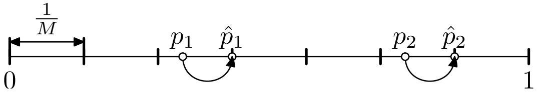
## Graph lemma

- Graph $G$ with $V(G) = \{1, \ldots, N\}$
- $d_i^-$ – in-degree of vertex $i$
- $\alpha$ – independence set of $G$
- Turán's Theorem + induction

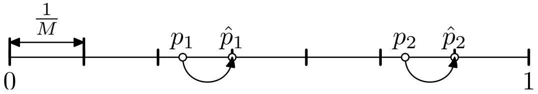
$$\sum_{i=1}^{N} \frac{1}{1 + d_i^-} \leq 2\alpha \log\left(1 + \frac{N}{\alpha}\right)$$

## Discretization




$$\sum_{i=1}^{N} \frac{p_{t,i}}{o_{t,i} + \gamma} = \sum_{i=1}^{N} \frac{p_{t,i}}{p_{t,i} + \sum_{j \in N_i^-} p_{t,j} + \gamma} \leq \sum_{i=1}^{N} \frac{\hat{p}_{t,i}}{\hat{p}_{t,i} + \sum_{j \in N_i^-} \hat{p}_{t,j}} + 2$$

## Discretization

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**Note:** we set $M = \lceil N^2/\gamma \rceil$

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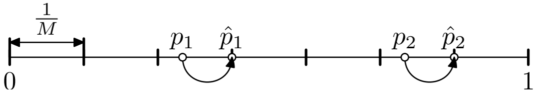


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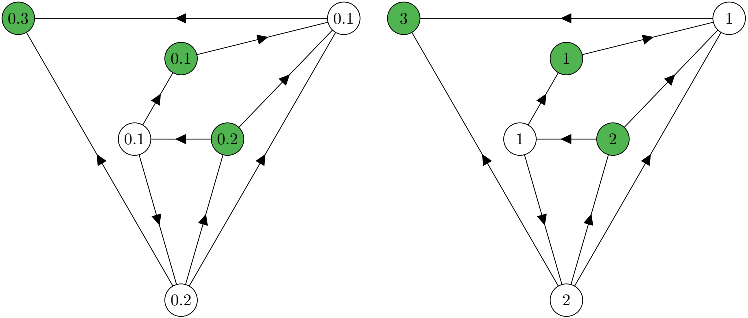
**Example:** let $M = 10$



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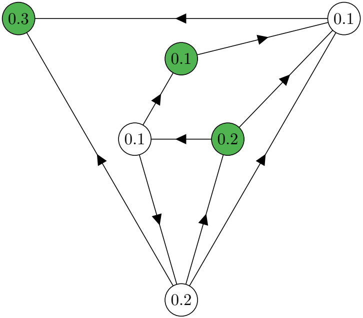

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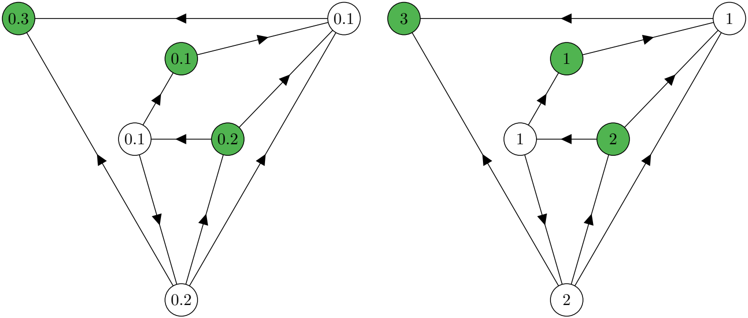

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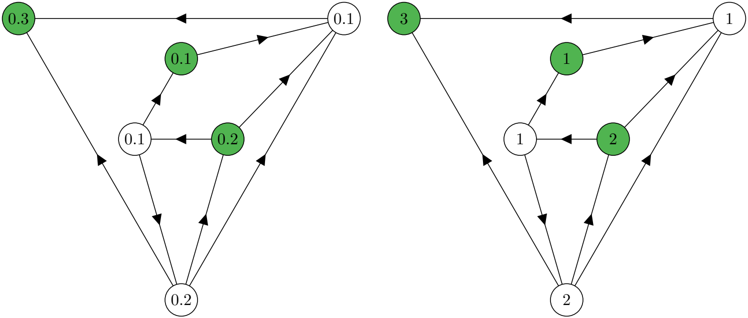

**Exp3-IX** regret bound

$$R_T \leq \frac{\log N}{\eta} + \left(\frac{\eta}{2} + \gamma\right) \sum_{t=1}^{T} \mathbb{E}\left[2\alpha_t \log\left(1 + \frac{\lceil N^2/\gamma \rceil + N}{\alpha_t}\right) + 2\right]$$

$$R_T = \widetilde{\mathcal{O}}\left(\sqrt{\overline{\alpha} T \log(N)}\right)$$

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# Next step

**Exp3-IX** regret bound

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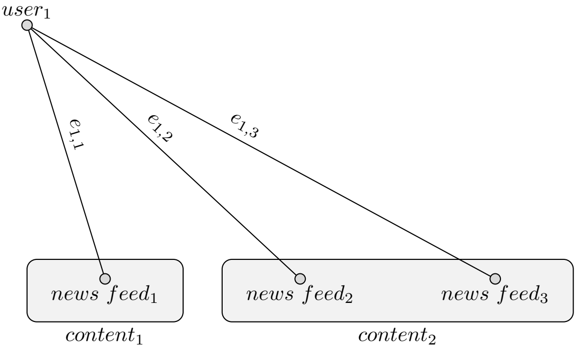
$$R_T = \widetilde{\mathcal{O}}\left(\sqrt{\overline{\alpha}\, T \log(N)}\right)$$

# Next step

Generalization of the setting to **combinatorial actions**

# Example


$user_1$
$e_{1,1}$
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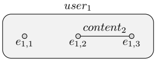
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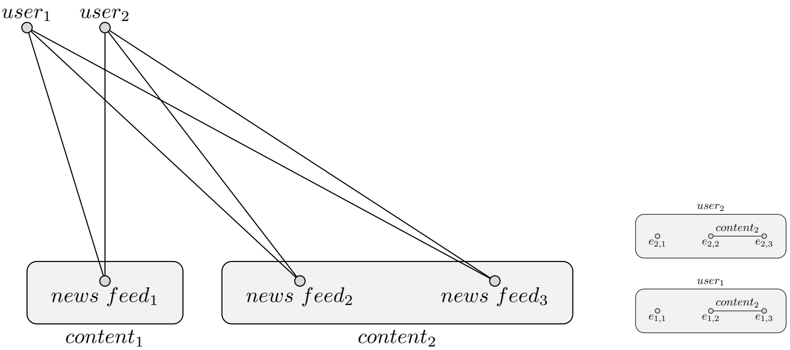
# Example



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$user_2$
$content_2$
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$content_2$
$e_{1,1}$
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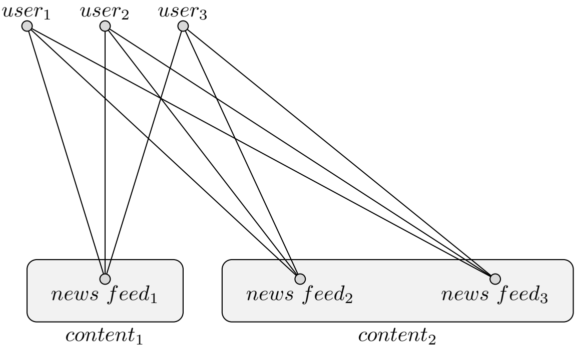
$e_{1,3}$

# Example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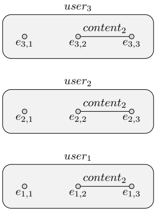
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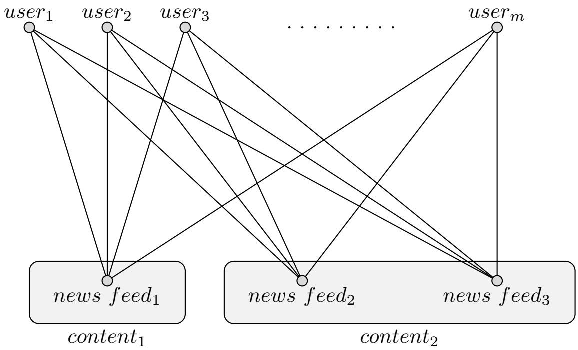
# Example



$user_1$
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$news\ feed_1$
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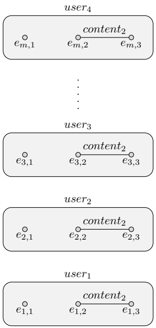
$content_1$
$content_2$

$user_4$
$content_2$
$e_{m,1}$
$e_{m,2}$
$e_{m,3}$
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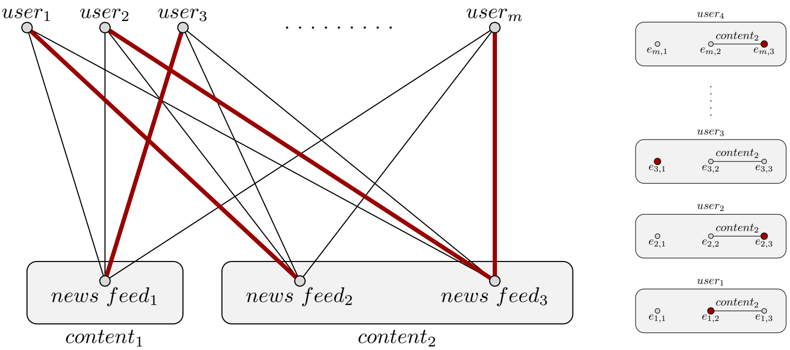
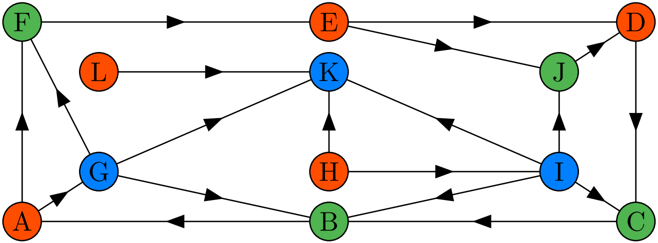
# Example




- Play $m$ out of $N$ nodes (combinatorial structure)
- Obtain losses of all played nodes
- Observe losses of all neighbors of played nodes




- Play action $\mathbf{V}_t \in S \subset \{0,1\}^N$, $\|\mathbf{v}\|_1 \leq m$ fro all $\mathbf{v} \in S$
- Obtain losses $\mathbf{V}_t^\mathsf{T} \ell_t$
- Observe additional losses according to the graph

## FPL-IX algorithm

- **Draw perturbation** $Z_{t,i} \sim \mathrm{Exp}(1)$ for all $i \in [N]$

- **Play "the best" action** $\mathbf{V}_t$ according to total loss estimate $\widehat{\mathbf{L}}_{t-1}$ and perturbation $\mathbf{Z}_t$

$$\mathbf{V}_t = \arg\min_{\mathbf{v}\in\mathcal{S}} \mathbf{v}^\mathsf{T} \left(\eta_t \widehat{\mathbf{L}}_{t-1} - \mathbf{Z}_t\right)$$

- **Compute loss estimates**

$$\hat{\ell}_{t,i} = \ell_{t,i} K_{t,i} \mathbb{1}\{\ell_{t,i} \text{ is observed}\}$$

- $K_{t,i}$: geometric random variable with

$$\mathbb{E}\left[K_{t,i}\right] = \frac{1}{o_{t,i} + (1 - o_{t,i})\gamma}$$

## **FPL-IX** - regret bound

$$R_T = \widetilde{\mathcal{O}}\left(m^{3/2}\sqrt{\sum_{t=1}^{T}\alpha_t}\right) = \widetilde{\mathcal{O}}\left(m^{3/2}\sqrt{\overline{\alpha}T}\right)$$

## Side Observation Summary

- **I**mplicit e**X**ploration idea
- **New algorithm for simple actions - Exp3-IX**
  - Using implicit exploration idea
  - Same regret bound as previous algorithm
  - No need to know graph before an action is played
  - Computationally efficient
- **New combinatorial setting with side observations**
- **Algorithm for combinatorial setting - FPL-IX**
- **Future directions**
  - No need to know graph after an action is played
  - Stochastic side observations
  - Random graph models
  - Exploiting the communities

**Michal Valko**
michal.valko@inria.fr
sequel.lille.inria.fr

**Sylvester's determinant theorem:**

$$|\mathbf{A} + \mathbf{x}\mathbf{x}^\intercal| = |\mathbf{A}||\mathbf{I} + \mathbf{A}^{-1}\mathbf{x}\mathbf{x}^\intercal| = |\mathbf{A}|(1 + \mathbf{x}^\intercal\mathbf{A}^{-1}\mathbf{x})$$

**Goal:**

- Upperbound determinant $|\mathbf{A} + \mathbf{x}\mathbf{x}^\intercal|$ for $\|\mathbf{x}\|_2 \leq 1$
- Upperbound $\mathbf{x}^\intercal\mathbf{A}^{-1}\mathbf{x}$

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**Goal:**

- Upperbound determinant $|\mathbf{A} + \mathbf{x}\mathbf{x}^\intercal|$ for $\|\mathbf{x}\|_2 \leq 1$
- Upperbound $\mathbf{x}^\intercal\mathbf{A}^{-1}\mathbf{x}$

$$\mathbf{x}^\intercal\mathbf{A}^{-1}\mathbf{x} = \mathbf{x}^\intercal\mathbf{Q}\boldsymbol{\Lambda}^{-1}\mathbf{Q}^\intercal\mathbf{x} = \mathbf{y}^\intercal\boldsymbol{\Lambda}^{-1}\mathbf{y} = \sum_{i=1}^{N}\lambda_i y_i^2$$

**Sylvester's determinant theorem:**

$$|\mathbf{A} + \mathbf{x}\mathbf{x}^\mathsf{T}| = |\mathbf{A}||\mathbf{I} + \mathbf{A}^{-1}\mathbf{x}\mathbf{x}^\mathsf{T}| = |\mathbf{A}|(1 + \mathbf{x}^\mathsf{T}\mathbf{A}^{-1}\mathbf{x})$$

**Goal:**

- Upperbound determinant $|\mathbf{A} + \mathbf{x}\mathbf{x}^\mathsf{T}|$ for $\|\mathbf{x}\|_2 \leq 1$
- Upperbound $\mathbf{x}^\mathsf{T}\mathbf{A}^{-1}\mathbf{x}$

$$\mathbf{x}^\mathsf{T}\mathbf{A}^{-1}\mathbf{x} = \mathbf{x}^\mathsf{T}\mathbf{Q}\boldsymbol{\Lambda}^{-1}\mathbf{Q}^\mathsf{T}\mathbf{x} = \mathbf{y}^\mathsf{T}\boldsymbol{\Lambda}^{-1}\mathbf{y} = \sum_{i=1}^{N} \lambda_i y_i^2$$

- $\|\mathbf{y}\|_2 \leq 1$.
- $\mathbf{y}$ is a canonical vector.
- $\mathbf{x} = \mathbf{Q}\mathbf{y}$ is an eigenvector of $\mathbf{A}$.

**Corollary**:
Determinant $|\mathbf{V}_T|$ of $\mathbf{V}_T = \mathbf{\Lambda} + \sum_{t=1}^T \mathbf{x}_t\mathbf{x}_t^\mathsf{T}$ is maximized when all $\mathbf{x}_t$ are aligned with axes.

$$|\mathbf{V}_T| \leq \max_{\sum t_i = T} \prod(\lambda_i + t_i)$$

$$\ln \frac{|\mathbf{V}_T|}{|\mathbf{\Lambda}|} \leq \max_{\sum t_i = T} \sum \ln\left(1 + \frac{t_i}{\lambda_i}\right)$$

$$\ln \frac{|\mathbf{V}_T|}{|\mathbf{\Lambda}|} \leq \sum_{i=1}^{d} \ln\left(1 + \frac{T}{\lambda}\right) + \sum_{i=d+1}^{N} \ln\left(1 + \frac{t_i}{\lambda_{d+1}}\right)$$

$$\leq d \ln\left(1 + \frac{T}{\lambda}\right) + \frac{T}{\lambda_{d+1}}$$

$$\leq 2d \ln\left(1 + \frac{T}{\lambda}\right)$$

$$\boldsymbol{f}^\mathsf{T}\mathcal{L}\boldsymbol{f} = \frac{1}{2}\sum_{i,j\leq N} w_{i,j}(f_i - f_j)^2 = S_G(f)$$

**Proof:**

$$\begin{aligned}
\boldsymbol{f}^\mathsf{T}\mathcal{L}\boldsymbol{f} &= \boldsymbol{f}^\mathsf{T}\mathcal{D}\boldsymbol{f} - \boldsymbol{f}^\mathsf{T}\mathcal{W}\boldsymbol{f} = \sum_{i=1}^{N} d_i f_i^2 - \sum_{i,j\leq N} w_{i,j} f_i f_j \\
&= \frac{1}{2}\left(\sum_{i=1}^{N} d_i f_i^2 - 2\sum_{i,j\leq N} w_{i,j} f_i f_j + \sum_{j=1}^{N} d_i f_j^2\right) = \frac{1}{2}\sum_{i,j\leq N} w_{i,j}(f_i - f_j)^2
\end{aligned}$$

# SpectralUCB analysis sketch

- Derivation of the confidence ellipsoid for $\hat{\boldsymbol{\alpha}}$ with probability $1-\delta$.
  - Using analysis of OFUL (Abbasi-Yadkori et al., 2011)

$$|x^\mathsf{T}(\hat{\boldsymbol{\alpha}} - \boldsymbol{\alpha}^*)| \leq \|x\|_{\mathbf{V}_t^{-1}} \left( R\sqrt{2\ln\left(\frac{|V_t|^{1/2}}{\delta|\boldsymbol{\Lambda}|^{1/2}}\right)} + C \right)$$

- Regret in one time step: $r_t = \mathbf{x}_*^\mathsf{T}\boldsymbol{\alpha}^* - \mathbf{x}_{\pi(t)}^\mathsf{T}\boldsymbol{\alpha}^* \leq 2c_t\|\mathbf{x}_{\pi(t)}\|_{\mathbf{V}_t^{-1}}$
- Cumulative regret:

$$R_T = \sum_{t=1}^T r_t \leq \sqrt{T\sum_{t=1}^T r_t^2} \leq 2(\ c_T + 1)\sqrt{2T\ln\frac{|\mathbf{V}_T|}{|\boldsymbol{\Lambda}|}}$$

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- Upperbound for $\ln(|\mathbf{V}_t|/|\boldsymbol{\Lambda}|)$

$$\ln\frac{|\mathbf{V}_t|}{|\boldsymbol{\Lambda}|} \leq \ln\frac{|\mathbf{V}_T|}{|\boldsymbol{\Lambda}|} \leq 2d\ln\left(\frac{\lambda+T}{\lambda}\right)$$

# SpectralTS analysis sketch

**Divide arms into two groups**

- $\Delta_i = \mathbf{b}_*^\mathsf{T}\boldsymbol{\mu} - \mathbf{b}_i^\mathsf{T}\boldsymbol{\mu} \leq g\|\mathbf{b}_i\|_{\mathbf{B}_t^{-1}}$ arm $i$ is **unsaturated**
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**Unsaturated arm**

- **Low regret** bounded by a factor of standard deviation
- **High probability** of picking

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  - Using analysis of OFUL algorithm (Abbasi-Yadkori et al., 2011)

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- Concentration of sample $\tilde{\boldsymbol{\mu}}$ around mean $\hat{\boldsymbol{\mu}}$ (with probability $1 - 1/T^2$)
  - Using concentration inequality for Gaussian random variable.

$$|\mathbf{b}_i^{\mathsf{T}}\tilde{\boldsymbol{\mu}} - \mathbf{b}_i^{\mathsf{T}}\hat{\boldsymbol{\mu}}| \leq \left(R\sqrt{6d\log\left(\frac{\lambda + T}{\delta\lambda}\right)} + C\right)\|\mathbf{b}_i\|_{\mathbf{B}_t^{-1}}\sqrt{4\log(TN)} = v\|\mathbf{b}_i\|_{\mathbf{B}_t^{-1}}\sqrt{4\log(TN)}$$

# SpectralTS analysis sketch

**Define** $\text{regret}'(t) = \text{regret}(t) \cdot \mathbb{1}\{|\mathbf{b}_i^\mathsf{T}\hat{\boldsymbol{\mu}}(t) - \mathbf{b}_i^\mathsf{T}\boldsymbol{\mu}| \le \ell \|\mathbf{b}_i\|_{\mathbf{B}_t^{-1}}\}$

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$(Y_t;\ t = 0, \ldots, T)$ is a **super-martingale** process w.r.t. history $\mathcal{F}_t$.

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# Backup: SpectralEliminator pseudocode

**Input:**

$N$ : the number of nodes, $T$ : the number of pulls

$\{\mathbf{\Lambda}_{\mathcal{L}}, \mathbf{Q}\}$ spectral basis of $\mathcal{L}$

$\lambda$ : regularization parameter

$\beta, \{t_j\}_j^J$ parameters of the elimination and phases

$A_1 \leftarrow \{\mathbf{x}_1, \ldots, \mathbf{x}_K\}$.

**for** $j = 1$ **to** $J$ **do**

$\mathbf{V}_{t_j} \leftarrow \gamma \mathbf{\Lambda}_{\mathcal{L}} + \lambda \mathbf{I}$

**for** $t = t_j$ **to** $\min(t_{j+1} - 1, T)$ **do**

Play $\mathbf{x}_t \in A_j$ with the largest width to observe $r_t$:

$\mathbf{x}_t \leftarrow \arg\max_{\mathbf{x} \in A_j} \|\mathbf{x}\|_{\mathbf{V}_t^{-1}}$

$\mathbf{V}_{t+1} \leftarrow \mathbf{V}_t + \mathbf{x}_t \mathbf{x}_t^\mathsf{T}$

**end for**

Eliminate the arms that are not promising:

$\hat{\boldsymbol{\alpha}}_t \leftarrow \mathbf{V}_t^{-1}[\mathbf{x}_{t_j}, \ldots, \mathbf{x}_t][r_{t_j}, \ldots, r_t]^\mathsf{T}$

$A_{j+1} \leftarrow \left\{ \mathbf{x} \in A_j, \langle \hat{\boldsymbol{\alpha}}_t, \mathbf{x} \rangle + \|\mathbf{x}\|_{V_t^{-1}} \beta \geq \max_{\mathbf{x} \in A_j} \left[ \langle \hat{\boldsymbol{\alpha}}_t, \mathbf{x} \rangle - \|\mathbf{x}\|_{\mathbf{V}_t^{-1}} \beta \right] \right\}$

**end for**

# Backup: SpectralEliminator analysis

**SpectralEliminator**

- Divide time into sets $(t_1 = 1 \leq t_2 \leq \dots)$ to introduce independence for Azuma-Hoeffding inequality and observe $R_T \leq \sum_{j=0}^{J}(t_{j+1} - t_j)\big[\langle \mathbf{x}^* - \mathbf{x}_t, \hat{\alpha}_j\rangle + (\|\mathbf{x}^*\|_{\mathbf{V}_j^{-1}} + \|\mathbf{x}_t\|_{\mathbf{V}_j^{-1}})\beta\big]$
- Bound $\langle \mathbf{x}^* - \mathbf{x}_t, \hat{\alpha}_j\rangle$ for each phase
- No bad arms: $\langle \mathbf{x}^* - \mathbf{x}_t, \hat{\alpha}_j\rangle \leq (\|\mathbf{x}^*\|_{\mathbf{V}_j^{-1}} + \|\mathbf{x}_t\|_{\mathbf{V}_j^{-1}})\beta$
- By algorithm: $\|\mathbf{x}\|^2_{\mathbf{V}_j^{-1}} \leq \frac{1}{t_j - t_{j-1}} \sum_{s=t_{j-1}+1}^{t_j} \|\mathbf{x}_s\|^2_{\mathbf{V}_{s-1}^{-1}}$
- $\sum_{s=t_{j-1}+1}^{t_j} \min\left(1, \|\mathbf{x}_s\|^2_{\mathbf{V}_{s-1}^{-1}}\right) \leq \log \frac{|\mathbf{V}_j|}{|\Lambda|}$