Optimistic Optimization of a Brownian Motion

**Algorithms**

**Algorithm 1 OOB algorithm**

1. **Input:** \( \varepsilon \)
2. **Init:** \( I \leftarrow \{[0,1]\}, t_1 = W(1) \)
3. for \( i = 2, 3, 4, \ldots \) do
4. \( [a, b] \in \arg \max_{I \subseteq B} \{ \text{break ties arbitrarily} \} \)
5. if \( \eta_t(b-a) \leq \varepsilon \) then break
6. end if
7. \( t_i \leftarrow W\left(\frac{a+b}{2}\right) \)
8. \( I \leftarrow I \cup [a, \frac{a+b}{2}] \cup [\frac{a+b}{2}, b] \setminus \{a, b\} \)
9. end for
10. **Output:** location \( \tilde{t}_\varepsilon \leftarrow \arg \max_{t \in I} W(t) \) and its value \( W(\tilde{t}_\varepsilon) \)

**Guarantees**

**THEOREM:** For any \( \varepsilon < 1/2 \)

\[
\mathbb{P}[M - W(\tilde{t}_\varepsilon) > \varepsilon] \leq \varepsilon
\]

\[
\mathbb{E}[N_\varepsilon] \leq c \log^2(1/\varepsilon)
\]

**Proof**

- 1. Correctness: algorithm definition + the law of Brownian bridge
- 2. at OOB evaluates pretty much only near-optimal points
- Denisov (1984): rewrite the motion as two Brownian meanders
- By Durett et al. (1977) the expected number of near-optimal points is bounded as \( \mathbb{E}[X_n(o)] \leq 6n^{1/2} \) which is \( O(\log(1/\varepsilon)) \)

**Open problem**

- Munos (2011) classifies functions according to \((d, C)\) to:
  - easy, \( d = 0 \), exponentially fast optimization
  - difficult, \( d \geq 0 \), polynomially fast optimization
- Open questions for a Brownian process:
  - what is its dimension \( d \)
  - how fast can we optimize it

**Challenge:** Brownian motion is a stochastic process!

**Our answers that solve the open problem:**

- \( \forall x, \text{w.p.} 1-\varepsilon, W(t) = t \) }.Lipschitz + \( \exists C(t) \) s.t. \( \text{Brownian} \in (d, C(t)) \)
- there is no \((d, C)\) with \( C \leq \infty \) such that \( \text{Brownian} \in (d, C) \)
- we can optimize it with sample complexity of \( O(\log^2(1/\varepsilon)) \)