An upgrade of a linear PID controller to a homogeneous one with application to quadrotor control

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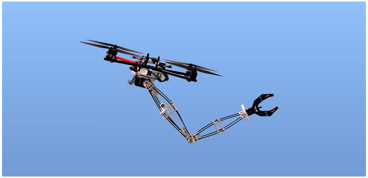
Application of quadrotor



Application: Rescue, Transportation, Monitor, Operation

Quadrotor controller

1. Linear controller
   - PID [Bouabdallah et al., 2004][Li and Li, 2011]
   - Linear quadratic regulator [Minh and Ha, 2010][Reyes-Valeria et al., 2013]
   - Gain-scheduling [Ataka et al., 2013]
2. Non-linear controller
   - Feedback linearization [Mokhtari et al., 2005][Lee et al., 2009]
   - Backstepping control [Bouabdallah and Siegwart, 2005][Madani, 2006]
   - Model predictive control [Alexis et al., 2012][Bangura and Mahony, 2014]
   - Sliding mode control [Wang et al., 2017]

Why homogeneous controller?

1. Improve the control performance without the peaking effect
2. Higher precision and finite-time stable without the chattering problem
3. Relative simple controller adaptive to the on-board calculation
4. More robust than linear PID controller
5. **Easy to implement the homogenization of PID controller based on the given PID parameters**, which is potential for many practical cases.

Objectives

Build a Homogeneous controller based on the linear controller gains to realize the faster and finite-time stabilization.

- Linear controller

$$u(x) = K_{lin}x \quad K_{lin} \in \mathbb{R}^{m\times n}, x \in \mathbb{R}^n \tag{1}$$

- Homogeneous controller

$$u(x) = K_0 x + |x|_{\mathbf{d}}^{1+\mu} K\mathbf{d}(-\ln|x|_{\mathbf{d}})x \tag{2}$$

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Homogeneity in physics

Homogeneity is a kind of symmetry with respect to dilation.

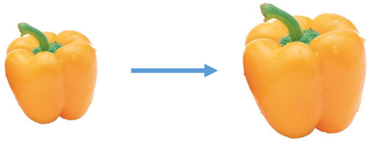

Figure: Invariant shape after dilation

Classical homogeneity

Leonhard Euler introduced the standard homogeneity in 18^{th}.

Definition 1.

Let n and m be two positive integers and $x \mapsto \lambda x$ be dilation. A mapping $f : \mathbb{R}^n \mapsto \mathbb{R}^m$ is said to be homogeneous with degree $\kappa \in \mathbb{R}$ in the classical sense iff

$$\forall \lambda > 0 : f(\lambda x) = \lambda^\kappa f(x) \tag{3}$$

Example 2.

A polynomial function $f(x) = x_1^2 + x_1 x_2 + x_2^2$ is homogeneous of degree 2.

$$f(\lambda x) = \lambda^2 x_1^2 + \lambda^2 x_1 x_2 + \lambda^2 x_2^2 = \lambda^2 f(x)$$

Weighted homogeneity

Weighted dilation noted as

$$x \mapsto \Lambda x \tag{4}$$

is a linear mapping $\mathbb{R}^n \mapsto \mathbb{R}^n$ where $\boldsymbol{r}$ is the generalized weights.

$$\Lambda = \begin{bmatrix} \lambda^{r_1} & 0 & 0 & \cdots & 0 \\ 0 & \lambda^{r_2} & 0 & \cdots & 0 \\ 0 & 0 & \lambda^{r_3} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \lambda^{r_n} \end{bmatrix} \tag{5}$$

Weighted homogeneity

Definition 3.

[zubov, 1958] Let $\boldsymbol{r}$ be a generalized weight, a function $f : \mathbb{R}^n \mapsto \mathbb{R}$ is said to be **r**-homogeneous of degree κ iff

$$f(\Lambda x) = \lambda^\kappa f(x), \quad \forall x \in \mathbb{R}^n, \quad \forall \lambda > 0 \tag{6}$$

Example 4.

A polynomial function

$$(x_1, x_2) \mapsto x_1^4 + x_1^2 x_2^4 + x_2^8 \tag{7}$$

is **r**-homogeneous of degree 8 with respect to weighted dilation $(x_1, x_2) \mapsto (\lambda^2 x_1, \lambda x_2)$

Weighted homogeneity

Definition 5.

[zubov, 1958] Let **r** be a generalized weight, a vector field f is said to be **r**-homogeneous with degree κ iff

$$f(\Lambda x) = \lambda^{\kappa} \Lambda f(x), \quad \forall x \in \mathbb{R}^n, \quad \forall \lambda > 0 \tag{8}$$

Note: A vector field is homogeneous of degrees κ in the classical sense (in Definition 1) iff it is **r**-homogeneous of degree $\kappa - 1$(in Definition 5).

Example 6.

The vector filed $\dot{x} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x$ is $[2, 1]$-homogeneous of degree -1.

Generalized homogeneity

Generalized dilation is defined as

$$x \to \mathbf{d}(s)x, \quad s \in \mathbb{R} \tag{9}$$

where

$$\mathbf{d}(s) = e^{G_{\mathbf{d}} s} = \sum_{i=0}^{+\infty} \frac{s^i G_{\mathbf{d}}^i}{i!}, \quad G_{\mathbf{d}} = \lim_{s \to 0} \frac{\mathbf{d}(s) - I}{s} \tag{10}$$

Generalized homogeneity

Definition 7.

[Kawski, 1991] A map $\boldsymbol{d} : \mathbb{R} \mapsto \mathbb{R}^{n\times n}$ is called **dilation** in $\mathbb{R}^n$ if it satisfies

- Group property: $\boldsymbol{d}(0) = I_n$ and $\boldsymbol{d}(t+s) = \boldsymbol{d}(t)\boldsymbol{d}(s) = \boldsymbol{d}(s)\boldsymbol{d}(t), \forall t, s \in \mathbb{R}$;
- Continuity property: $\boldsymbol{d}(s)$ is continuous map, i.e.

$$\forall t, \epsilon > 0, \exists \sigma > 0 : |s-t| < \sigma \Rightarrow |\boldsymbol{d}(s) - \boldsymbol{d}(t)|_A \leq \epsilon$$

- Limit property: $\lim_{s\to-\infty} |\boldsymbol{d}(s)x| = 0$ and $\lim_{s\to+\infty} |\boldsymbol{d}(s)x| = +\infty$.

Generalized homogeneity

Generalized dilation $\mathbf{d}$ should satisfy all the properties in Definition 7.

Example 8.

- Uniform dilation

$$\mathbf{d}_1(s) = e^s I_n, \quad s \in \mathbb{R}, \quad G_{\mathbf{d}} = I_n \tag{11}$$

- weighted dilation [zubov, 1958]

$$\mathbf{d}_2(s) = \begin{bmatrix} e^{r_1 s} & 0 & \cdots & 0 \\ 0 & e^{r_2 s} & \cdots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \cdots & e^{r_n s} \end{bmatrix} \quad s \in \mathbb{R}, \tag{12}$$

with $G_{\mathbf{d}} = diag\{r_i\}, r_i > 0, \quad i = 1, 2, ..., n.$

Generalized homogeneity

Definition 9.

The dilation $\mathbf{d}$ is **strictly monotone** if $\exists \beta > 0$ such that

$$\|\mathbf{d}(s)\| \leq e^{\beta s}, \quad \forall s \leq 0. \tag{13}$$

Generalized homogeneity

Theorem 10.

[Polyakov, 2018] Let $\mathbf{d}$ *be a dilation in the Euclidean space* $\mathbb{R}^n$ *with the inner product*

$$\langle u, v\rangle = u^\top P v, \quad u, v \in \mathbb{R}^n,$$

where $0 \prec P = P^\top \in \mathbb{R}^{n\times n}$ *is a positive definite symmetric matrix. The dilation* $\mathbf{d}$ *is strictly monotone in* $\mathbb{R}^n$ *equipped with the norm* $\|z\| = \sqrt{\langle z, z\rangle}$ *if and only if the following linear matrix inequality holds*

$$PG_{\mathbf{d}} + G_{\mathbf{d}}^\top P \succ 0, \quad P \succ 0 \tag{14}$$

where $G_{\mathbf{d}} \in \mathbb{R}^n$ *is the generator of the dilation* $\mathbf{d}$

Canonical homogeneous norm

Definition 11.

The function $\|\cdot\|_{\mathbf{d}} : \mathbb{R}^n \backslash \{0\} \to (0, +\infty)$ defined as

$$\|x\|_{\mathbf{d}} = e^{s_x}, \text{ where } s_x \in \mathbb{R} : \|\mathbf{d}(-s_x)x\| = 1, \quad (15)$$

is called the canonical homogeneous norm, where $\mathbf{d}$ is a strictly monotone dilation.

In this presentation, we always use following norm

$$\|\mathbf{d}(-s_x)x\| = \sqrt{x^\top \mathbf{d}^\top(-s_x) P \mathbf{d}(-s_x) x}$$

.

Monotonicity of dilation

Theorem 12.

[Polyakov et al., 2018] If $\mathbf{d}$ *is a strictly monotone continuous dilation on* $\mathbb{R}^n$ *then*

- *the function* $\|\cdot\|_{\mathbf{d}} : \mathbb{R}^n \backslash \{0\} \to \mathbb{R}_+$ *given by* (15) *is* **single-valued and positive**;
- $\|x\|_{\mathbf{d}} \to 0$ *as* $x \to 0$;
- *if the norm in* $\mathbb{R}^n$ *is defined as* $\|x\| = \sqrt{x^\top P x}$ *with* $P \in \mathbb{R}^{n \times n}$ *satisfying* (14) *then*

$$\frac{\partial \|x\|_{\mathbf{d}}}{\partial x} = \|x\|_{\mathbf{d}} \frac{x^\top \mathbf{d}^\top(-\ln \|x\|_{\mathbf{d}}) P \mathbf{d}(-\ln \|x\|_{\mathbf{d}})}{x^\top \mathbf{d}^\top(-\ln \|x\|_{\mathbf{d}}) P G_{\mathbf{d}} \mathbf{d}(-\ln \|x\|_{\mathbf{d}}) x} \tag{16}$$

for any $x \neq 0$.

$\|x\|_{\mathbf{d}}$ is going to be considered as a Lyapunov function candidate.

Homogeneous systems

Definition 13.

[Kawski, 1991] A vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ is said to be $\mathbf{d}$-homogeneous of degree $\nu \in \mathbb{R}$ if

$$f(\mathbf{d}(s)x) = e^{\nu s}\mathbf{d}(s)f(x), \quad \text{for} \quad s \in \mathbb{R}, \quad x \in \mathbb{R}^n \setminus \{0\} \tag{17}$$

Remark that a vector field $x \to Ax$ with $A \in \mathbb{R}^{n\times n}, x \in \mathbb{R}^n$ is $\mathbf{d}$-homogeneous of degree $\nu \in \mathbb{R}$ if and only if [Polyakov, 2019]

$$AG_{\mathbf{d}} = (\nu I + G_{\mathbf{d}})A \tag{18}$$

where $G_{\mathbf{d}} \in \mathbb{R}^{n\times n}$ is a generator of $\mathbf{d}$

Homogeneous systems

Proposition 2.1.

[Nakamura et al., 2002] If the system $\dot{\xi} = f(\xi)$ *is* $\mathbf{d}$*-homogeneous of degree* $\nu \in \mathbb{R}$ *and its origin is locally uniformly asymptotically stable then*

- *for* $\nu < 0$ *it is globally uniformly finite-time stable;*
- *for* $\nu = 0$ *it is globally uniformly asymptotically stable;*
- *for* $\nu > 0$ *it is globally uniformly nearly fixed-time stable, i.e.* $\forall r > 0$, $\exists T = T(r) > 0$: $\|x_{x_0}(t)\| < r$, $\forall t \geq T$, $\forall x_0 \in \mathbb{R}^n$.

Homogeneous stabilization of linear MIMO systems

Consider linear control system

$$\dot{x} = Ax + Bu(x), \quad t > 0, \tag{19}$$

where $x(t) \in \mathbb{R}^n$ is the system state, $u : \mathbb{R}^n \to \mathbb{R}^m$ is the feedback control, $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$ are system matrices.

Definition 14.

A system (19) is said to be $\mathbf{d}$*-homogeneously stabilizable with degree* $\mu \in \mathbb{R}$ *if there exists a bounded feedback law* $u : \mathbb{R}^n \to \mathbb{R}^m$ *such that the closed-loop system is globally asymptotically stable and* $\mathbf{d}$*-homogeneous of degree* μ

Homogeneous PD controller

Theorem 15.

If system (19) is controllable then homogeneous controller can be selected as

$$u(x) = K_0 x + \|x\|_{\mathbf{d}}^{1+\mu} Y X^{-1} \mathbf{d}(-\ln \|x\|_{\mathbf{d}}) x \qquad (20)$$

with any $K_0 \in \mathbb{R}^{n\times m}$ *such that* $A_0 = A + BK_0$ *is nilpotent,* $\mu \in [-1, k^{-1}]$, $k \leq n$, $\mathbf{d}$ *is generated by* $G_{\mathbf{d}} \in \mathbb{R}^{n\times n}$ *satisfying*

$$A_0 G_{\mathbf{d}} = (G_{\mathbf{d}} + \mu I) A_0, \quad G_{\mathbf{d}} B = B \qquad (21)$$

and $X \in \mathbb{R}^{n\times n}$, $Y \in \mathbb{R}^{m\times n}$ *satisfy*

$$\begin{cases} XA_0^\top + A_0 X + Y^\top B^\top + BY + XG_{\mathbf{d}}^\top + G_{\mathbf{d}} X = 0, \\ XG_{\mathbf{d}}^\top + G_{\mathbf{d}} X \succ 0, \quad X \succ 0, \end{cases} \qquad (22)$$

Homogeneous PID controller

Theorem 16.

Let $K_0 \in \mathbb{R}^{m\times n}$ be such that $A+BK_0$ is nilpotent and an anti-Hurwitz matrix $G_{\mathbf{d}} \in \mathbb{R}^{n\times n}$ satisfy (21). Let $X \in \mathbb{R}^{n\times n}$ and $Y \in \mathbb{R}^{m\times n}$ satisfy (22) then for any positive definite matrix $Q \in \mathbb{R}^{m\times m}$ the control law

$$u(x) = K_0 x + u_h(x) + \int_0^t u_I(x(s))ds, \qquad (23)$$

with $u_h = \|x\|_{\mathbf{d}}^{1/2} Y X^{-1} z$, $u_I = \frac{-QB^\top Pz}{z^\top PG_{\mathbf{d}} z}$, $z = \mathbf{d}(-\ln\|x\|_{\mathbf{d}})x$ stabilizes the origin of the system $\dot{x} = Ax + B(u(x)+p)$, in a finite-time time for any constant vector $p \in \mathbb{R}^m$.

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Process of upgrade linear controller

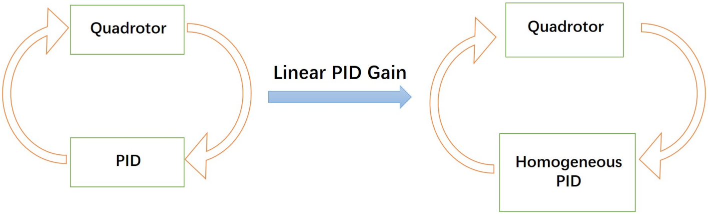

Figure: upgrade methodology

Linear feedback control

System (19) with linear controller is in the following form

$$\dot{x} = Ax + Bu_{lin}(x), \quad t > 0, \tag{24}$$

$$u_{lin} = K_{lin}x \tag{25}$$

where $x(t) \in \mathbb{R}^n$ is the system state, $u_{lin} : \mathbb{R}^n \to \mathbb{R}^m$ is the feedback control, $K_{lin} \in \mathbb{R}^{m\times n}$ be such that the matrix $A + BK_{lin}$ is Hurwitz, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ are system matrices

Homogenization of linear feedback

Corollary 17.

Let the pair $\{A, B\}$ *be controllable,* $K_0 \in \mathbb{R}^{m\times n}$ *make* $A_0 = A + BK_0$ *nilpotent,* $K_{lin} \in \mathbb{R}^{m\times n}$ *be given by Eq.*(25), $G_{\mathbf{d}} \in \mathbb{R}^{n\times n}$ *satisfies* (21) *for* $\mu = -1$ *and* $P = P^\top \in \mathbb{R}^{n\times n}$ *satisfies*

$$\begin{array}{l} (A + BK_{lin})^\top P + P(A + BK_{lin}) \prec 0 \\ G_{\mathbf{d}}^\top P + PG_{\mathbf{d}} \succ 0, \quad P \succ 0 \end{array} \tag{26}$$

then the control u *given by* (20) *with* $\mu = -1$ *and* $K = K_{lin} - K_0$ $\mathbf{d}$*-homogeneously stabilizes the origin of the system* (19) *in a finite-time. Moreover,* $u_{lin}(x) = u(x)$ *for* $x \in S = \{x \in \mathbb{R}^n : \|x\| = 1\}$.

Controller with saturation

Let the saturation function $\mathrm{sat}_{a,b} : \mathbb{R}_+ \to \mathbb{R}_+$ be given by

$$\mathrm{sat}_{a,b}(\rho) = \begin{cases} b & \text{if} \quad \rho \geq b, \\ \rho & \text{if} \quad a < \rho < b, \\ a & \text{if} \quad \rho < a, \end{cases} \qquad \rho \in \mathbb{R}_+. \tag{27}$$

controller with saturation is defined as

$$u_{a,b}(x) = K_0 x + K\mathbf{d}(-\ln \mathrm{sat}_{a,b}(\|x\|_{\mathbf{d}}))x, \tag{28}$$

From (27), we have

$$u_{1,1}(x) = K_{lin}x \text{ and } u_{0,+\infty}(x) = K_0 x + K\mathbf{d}(-\ln \|x\|_{\mathbf{d}})x. \tag{29}$$

Linear controller and homogeneous controller coincides on the unit sphere $x^\top P x = 1$.

Digital realization

Theorem 18.

If all conditions of Theorem 15 are fulfilled, then for any fixed $r > 0$ the closed $\mathbf{d}$-homogeneous ball $\|x\|_{\mathbf{d}} < r$ is a strictly positively invariant compact set[1] *of the closed-loop system* (19) *with the linear control*

$$u_r(x) = K_0 + r^{1+\mu} K \mathbf{d}(-\ln r)x. \tag{30}$$

[1] A set Ω is said to be a strictly positively invariant for a dynamical system if $x(t_0) \in \Omega \Rightarrow x(t) \in \Omega, t \geq t_0$, where x denotes a solution x of this system.

Digital realization

Corollary 19.

If

1. *all conditions of Theorem 15 are fulfilled;*
2. $\{t_i\}_{i=0}^{+\infty}$ *is an arbitrary sequence of time instances such that* $0 = t_0 < t_1 < t_2 < \dots$ *and* $\lim_{i\to+\infty} t_i = +\infty$;
3. u *is a linear switched control of the from*

$$u(x(t)) = \|x(t_i)\|_{\mathbf{d}}^{1+\mu} K \mathbf{d}(-\ln \|x(t_i)\|_{\mathbf{d}}) x(t), \; t \in [t_i, t_{i+1}) \quad (31)$$

then the closed-loop system (19) *is globally uniformly asymptotically stable.*

Algorithm to find $\|x\|_{\mathbf{d}}$

Algorithm 1.

Initialization $\underline{V} = a$; $\overline{V} = b$; $N_{\max} \in \mathbb{N}$;

if $x^\top(t_i)\mathbf{d}^\top(-\ln \overline{V})P\mathbf{d}(-\ln \overline{V})x(t_i) > 1$ *then*

$\underline{V} = \overline{V}$; $\overline{V} = \min(b, 2\overline{V})$;

elseif $x^\top(t_i)\mathbf{d}^\top(-\ln \underline{V})P\mathbf{d}(-\ln \underline{V})x(t_i) < 1$ *then*

$\overline{V} = \underline{V}$; $\underline{V} = \max(0.5\underline{V}, a)$;

else

  for $i = 1 : N_{\max}$

    $V = \frac{\underline{V}+\overline{V}}{2}$;

    if $x^\top(t_i)\mathbf{d}^\top(-\ln \underline{V})P\mathbf{d}(-\ln \underline{V})x(t_i) < 1$ *then*

      $\overline{V} = V$;

     else $\underline{V} = V$;

    endif;

  endfor;

endif;

$\|x(t_i)\|_{\mathbf{d}} \approx \overline{V}$;

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Quadrotor platform

- Platform under construction in Inria:
  6 cameras + one quadrotor + one PC

Q-Drone

Hovering

Quadrotor model

Quadrotor dynamics model could be simplified as following (32)

$$\dot{\zeta} = A\zeta + \begin{pmatrix} 0 \\ 0 \\ 0 \\ B \end{pmatrix} \begin{pmatrix} u_2 \\ u_3 \end{pmatrix}, \quad \ddot{\psi} = \frac{u_4}{I_{zz}}, \quad \ddot{z} = \frac{u_1}{m} \qquad (32)$$

where $u_1 = F_T - mg$, $u_2 = \tau_1$, $u_3 = \tau_2$, $u_4 = \tau_3$

$$A = \begin{pmatrix} 0 & E & 0 & 0 \\ 0 & 0 & gE & 0 \\ 0 & 0 & 0 & E \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} \frac{1}{I_{yy}} & 0 \\ 0 & \frac{1}{I_{xx}} \end{pmatrix}.$$

Note: the above model is simplified at the equilibrium point by smaller angle assumption and ignoring the Coriolis Force.

PID Controller

The PID controllers provided by Quanser are the following form:

$$u_1 = K_z \left(\begin{smallmatrix} z \\ \dot{z} \end{smallmatrix} \right) + \int K_I z dt, \quad \left(\begin{smallmatrix} u_2 \\ u_3 \end{smallmatrix} \right) = K_\zeta \zeta, \quad u_4 = K_\psi \begin{pmatrix} \psi \\ \dot{\psi} \end{pmatrix}$$

with the parameters (provided by the manufacturer)

$$K_\psi = \begin{bmatrix} -0.59 & 0.11 \end{bmatrix} \quad K_z = \begin{bmatrix} -35 & -14 \end{bmatrix}, K_I = -4$$
$$K_\zeta = \left(\begin{smallmatrix} -2.91 & 0 & -1.45 & 0 & -1.85 & 0 & -0.16 & 0 \\ 0 & -3.53 & 0 & -1.76 & 0 & -2.25 & 0 & -0.20 \end{smallmatrix} \right).$$

Homogeneous controller

- Given the gains of PID controller, homogeneous controller (20) and (23) can be applied to make the systems (32) be homogeneous of degree -1 with respect to dilation $\mathbf{d}_1(s) = \mathrm{diag}\{e^{4s}E, e^{3s}E, e^{2s}E, e^{1s}E\}$ and $\mathbf{d}_2(s) = \mathrm{diag}\{e^{2s}, e^{s}\}$
- System ξ is homogeneous of degree -1, with respect to dilation $\mathbf{d}_1(s)$
- System z and ψ are homogeneous of degree -1, with respect to dilation $\mathbf{d}_2(s)$

Experiment results

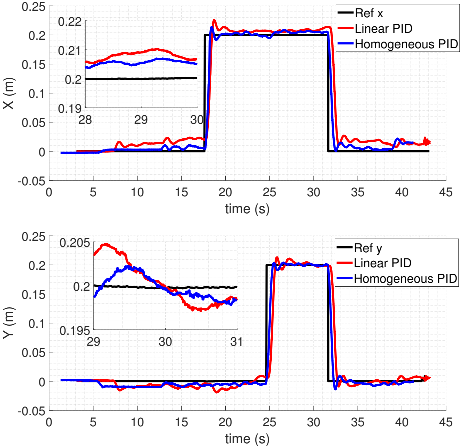

Figure: Quadrotor position tracking comparison of x and y

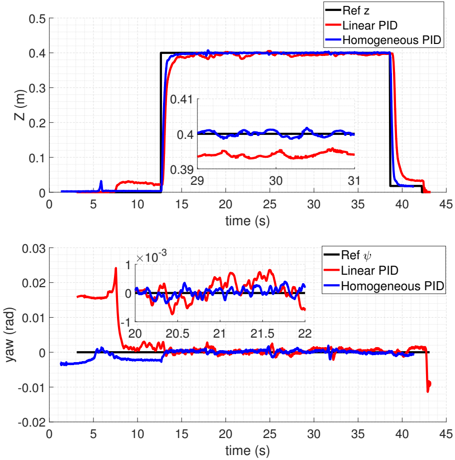

Figure: Quadrotor position tracking comparison of z and ψ

Least square error

Table: Mean value of stabilization error

L_2 Error (m)	Linear	Homogeneous	Improvement
$\|error_x\|_{L_2}$	0.0234	0.0138	41%
$\|error_y\|_{L_2}$	0.0081	0.0028	66%
$\|error_z\|_{L_2}$	0.0313	0.0071	77%
$\|error_\psi\|_{L_2}$	0.0036	0.0022	38%

Robustness

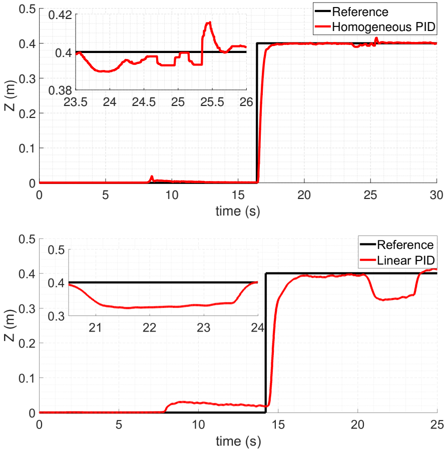

Figure: Quadrotor robustness comparison of PID and Homogeneous PID controller

Conclusion

Homogeneous controller

- Significantly improved the precision and robustness verified by experiments;
- Energy consuming is about $0.5 - 1\%$ more;
- Be easy to upgrade from a given linear PID controller;
- Be potential for many practical cases.

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Discrete-time differentiators:

implicit discretization and comparative analysis

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Introduction

First name:	Mohammad Rasool
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Birth date:	June 1988
Born in:	Isfahan, Iran

Education

PhD	Control Systems	University of Tabriz, Iran	2013 – 2017
MSc	Control Systems	IAU, Iran	2011 – 2013
BSc	Electrical Engineering	MAU, Iran	2007 – 2011

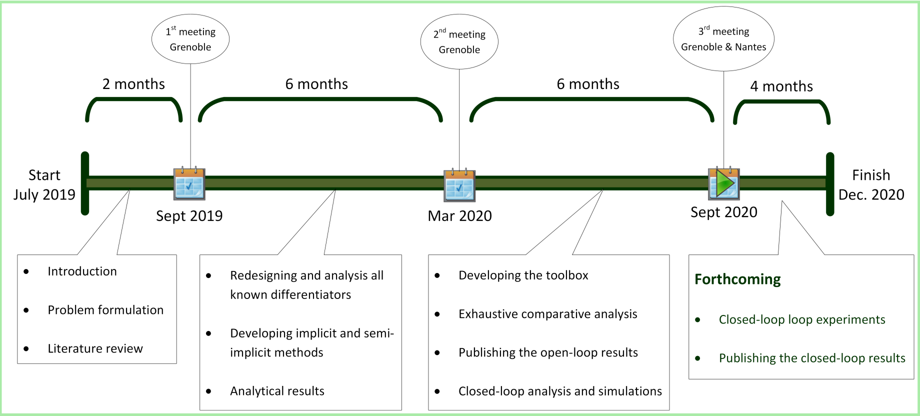
Experience

Postdoc	INRIA – Grenoble, France	2019 – 2020
Postdoc	University of Tabriz, Iran	2017 – 2019
Lecturer	University of Tabriz, Iran	2013 – 2017

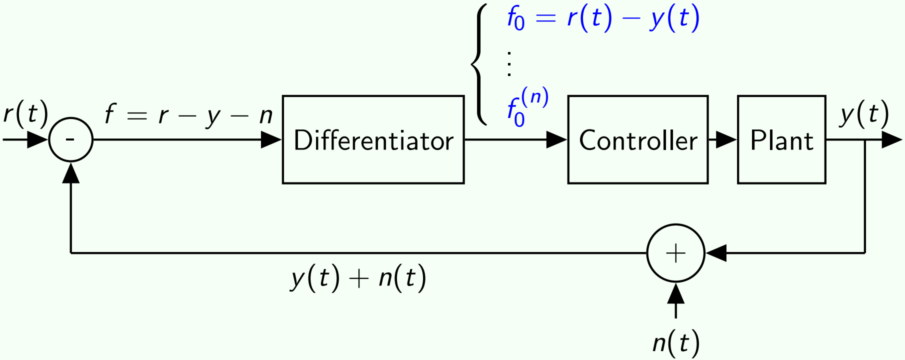
Research interests

nonlinear and hybrid control, implicit discretization, differentiation, ...

Project timeline



Differentiation in closed-loop control systems



$y(t)$: output

$n(t)$: noise

$r(t)$: reference

$f_0(t) = f(t) - n(t) = r(t) - y(t)$

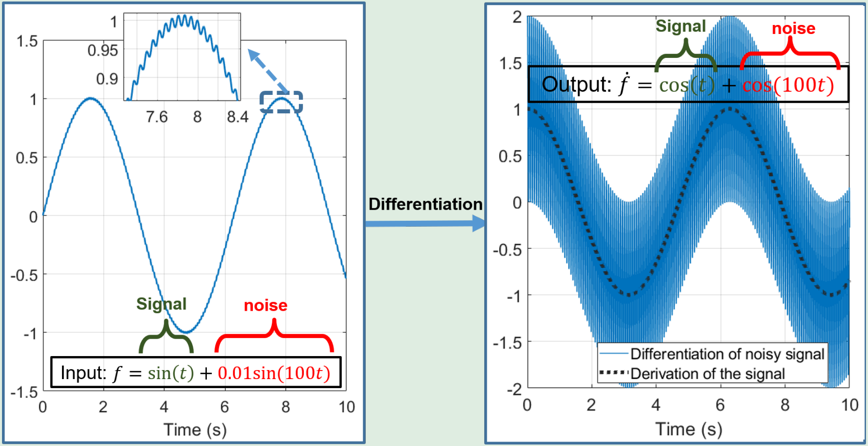
Definitions:

robustness to noise

robustness to disturbance

Challenges

Example: effect of the high-frequency noise on the differentiation



A conventional method

- Dirty filter:

$$\frac{Y(s)}{F(s)} = \underbrace{\frac{c}{s+c}}_{\text{LP filter}} s \qquad \begin{cases} \mathcal{L}\{f(t)\} = F(s) \\ \mathcal{L}\{y(t)\} = Y(s) \end{cases}$$

- For $c \to \infty$ it turns into the Euler differentiator:

$$\frac{Y(s)}{F(s)} = s$$

$y(t)$: output | $f(t) = \underbrace{f_0(t)}_{\text{signal}} + \underbrace{n(t)}_{\text{noise}}$: input

c: parameter | Drawbacks: phase-lag, difficult tuning, ...

Alternative methods (continuous-time differentiators)

1. Slotine-Hedrick-Misawa differentiator (SHMD)
2. Super-twisting differentiator (STD)
3. Arbitrary-order super-twisting differentiator super-twisting (AO-STD)
4. High-degree super-twisting differentiator (HD-STD)
5. Quadratic sliding-mode differentiator (QD)
6. Variable gain exponent differentiator (VGED)
7. Super-twisting differentiator with adaptive coefficients (STDAC)
8. ALgèbre pour Identification et Estimation Numériques (ALIEN)

Slotine-Hedrick-Misawa differentiator (SHMD)

- It's probably the first sliding-mode-based differentiator

$$\begin{cases} \dot{z}_i(t) \in z_{i+1}(t) - \alpha_i \Psi(\sigma_0(t)) - \kappa_i \sigma_0(t) \\ \dot{z}_n(t) \in -\alpha_n \Psi(\sigma_0(t)) - \kappa_n \sigma_0(t) \end{cases}$$

z_i: differentiation order i

n: order of the differentiator

$\Psi(\cdot)$: a set-valued function

$f(t) = \underbrace{f_0(t)}_{\text{signal}} + \underbrace{n(t)}_{\text{noise}}$: input

α_i, κ_i: parameters

$\sigma_0(t) = z_0(t) - f(t)$

$i = 0, 1, \ldots, n-1$

J.-J. E. Slotine, J. K. Hedrick, and E. A. Misawa, "On Sliding Observers for Nonlinear Systems" in JoDSMC., 1987

Super-twisting differentiator (STD)

- Homogeneous

$$\begin{cases} \dot{z}_0(t) = -\lambda_0 L^{\frac{1}{2}} \lceil \sigma_0(t) \rfloor^{\frac{1}{2}} + z_1(t) \\ \dot{z}_1(t) \in -\lambda_1 L \,\mathrm{sgn}(\sigma_0(t)) \end{cases}$$

z_1: first-order differentiation

$\sigma_0(t) = z_0(t) - f(t)$

$\lceil a \rfloor^b = |a|^b \,\mathrm{sgn}(a)$

λ_0, λ_1, L: parameters

$f(t)$: input

A. Levant, "Robust exact differentiation via sliding mode technique" in Automatica, 1998

Arbitrary-order super-twisting differentiator (AO-STD)

- The only discontinuous term only appears in the last row
- Homogeneous

$$\begin{cases} \dot{z}_i(t) = -\lambda_i L^{\frac{i+1}{n+1}} \lceil \sigma_0(t) \rfloor^{\frac{n-i}{n+1}} + z_{i+1}(t), \quad i = 0, \dots, n-1 \\ \dot{z}_n(t) \in -\lambda_n L \operatorname{sgn}(\sigma_0(t)), \end{cases}$$

z_i: differentiation order i

$\sigma_{0,k} = z_0(t) - f(t)$

n: order of the differentiator

$\lceil a \rfloor^b = |a|^b \operatorname{sgn}(a)$

λ_i, L: parameters

$f(t)$: input

$i = 0, 1, \dots, n-1$

A. Levant, "Higher-order sliding modes, differentiation and output-feedback control" in IJoC., 2003

High-degree super-twisting differentiator (HD-STD)

- Uniform convergence

$$\begin{cases} \dot{z}_0(t) = -\lambda_0 L^{\frac{1}{2}}\left(\lceil \sigma_0(t) \rfloor^{\frac{1}{2}} + \mu \lceil \sigma_0(t) \rfloor^{\frac{3}{2}}\right) + z_1(t) \\ \dot{z}_1(t) \in -\lambda_1 L\left(\frac{1}{2}\,\mathrm{sgn}(\sigma_0(t)) + 2\mu\sigma_0(t) + \frac{3}{2}\lceil \mu\sigma_0(t) \rfloor^{2}\right), \end{cases}$$

z_1: The first-order differentiation

$\sigma_0(t) = z_0(t) - f(t)$

$\lceil a \rfloor^{b} = |a|^{b}\,\mathrm{sgn}(a)$

$\lambda_0, \lambda_1, L, \mu$: parameters

$f(t)$: input

E. Cruz-Zavala, J. A. Moreno and L. M. Fridman, "Uniform Robust Exact Differentiator" in IEEE TAC., 2011

Quadratic sliding-mode differentiator (QD)

$$\begin{cases} \dot{z}_1(t) = z_2(t) \\ \dot{z}_2(t) \in \begin{cases} -\alpha F \operatorname{sgn}(\sigma_0(t)) & \text{if} \quad \sigma_0(t) z_2(t) > 0 \\ -F \operatorname{sgn}(\sigma_0(t)) & \text{if} \quad \sigma_0(t) z_2(t) < 0 \end{cases} \\ \sigma_0(t) = 2F(z_1(t) - f(t)) + |z_2(t)| z_2(t), \end{cases}$$

$z_1(t)$: Differentiation of $f(t)$	α, F, μ: parameters
$\sigma_0(t) = z_0(t) - f(t)$	$f(t)$: input
n: order of the differentiator	$\lceil a \rfloor^b = \|a\|^b \operatorname{sgn}(a)$

T. Emaru and T. Tsuchiya, "Research on estimating the smoothed value and the differential value..." in IEEE/RSJ., 2000

Variable gain exponent differentiator (VGED)

$$
\begin{cases}
\dot{z}_0(t) = -\lambda_0 \mu |\sigma_0(t)|^{\alpha(t)} \operatorname{sgn}(\sigma_0(t)) + z_1(t) \\
\dot{z}_1(t) = -\lambda_1 \alpha(t) \mu^2 |\sigma_0(t)|^{2\alpha(t)-1} \operatorname{sgn}(\sigma_0(t)) \\
\dot{\gamma}(t) = -\tau \gamma(t) + \tau |f_f(t)| \\
\alpha(t) = \frac{1}{2}\left(1 + \frac{\gamma^q}{\gamma^q + \epsilon}\right),
\end{cases}
$$

$$
f_f(t) = \mathcal{L}^{-1}\left\{ \frac{\left(\frac{s}{\omega_c}\right)^4 F(s)}{\left(\left(\frac{s}{\omega_c}\right)^2 + 0.7654\left(\frac{s}{\omega_c}\right) + 1\right)\left(\left(\frac{s}{\omega_c}\right)^2 + 1.8478\left(\frac{s}{\omega_c}\right) + 1\right)} \right\}
$$

$z_1(t)$: Differentiation of $f(t)$	$\lambda_0, \lambda_1, \mu, q, \epsilon, \tau$: parameters
$\sigma_0(t) = z_0(t) - f(t)$	$f(t)$: input
n: order of the differentiator	$\lceil a \rfloor^b = |a|^b \operatorname{sgn}(a)$

M. Ghanes and J. P. Barbot and L. Fridman and A. Levant and R. Boisliveau, "A New Varying Gain Exponent..." in TAC., 2020

STD with adaptive coefficients (STDAC)

$$\begin{cases} \dot{z}_0(t) = -\lambda_0\gamma(t)\lceil\sigma_0(t)\rfloor^{\frac{1}{2}} + z_1(t) \\ \dot{z}_1(t) \in -\lambda_1\gamma^2(t)\,\mathrm{sgn}(\sigma_0(t)) \end{cases}$$

$$\dot{\gamma}(t) = \frac{\gamma(t)}{2}\alpha \begin{cases} |\sigma_0(t)|^{-\frac{1}{2}} & \text{for} \quad |\sigma_0(t)| \geq 1 \\ |\sigma_0(t)| & \text{for} \quad |\sigma_0(t)| < 1 \\ \frac{1}{\gamma} - 1 & \text{for} \quad |\sigma_0(t)| < 1.1\epsilon, \end{cases}$$

$z_1(t)$: Differentiation of $f(t)$	$\lambda_0, \lambda_1, \alpha, \epsilon$: parameters
$\sigma_0(t) = z_0(t) - f(t)$	$f(t)$: input
n: order of the differentiator	$\lceil a\rfloor^b = \|a\|^b\,\mathrm{sgn}(a)$

M. Reichhartinger, S. Spurgeon, "An arbitrary-order differentiator design paradigm with adaptive gains" in IJoC., 2018

ALgèbre pour Identification et Estimation Numériques (ALIEN)

$$z^{(n)}(t) = \frac{(-1)^n \gamma_{\kappa,\mu,n}}{T^n} \int_0^1 \frac{d^n}{d\tau^n}\{\tau^{\kappa+n}(1-\tau)^{\mu+n}\} f(\tau T) d\tau$$

$$\gamma_{\kappa,\mu,n} = \frac{(\kappa+\mu+2n+1)!}{(\kappa+n)!(\mu+n)!}$$

$z^{(n)}(t)$: differentiation order n of the input	$f(t)$: input
n: order of the differentiator	κ, μ, T: Parameters

M. Mboup and S. Riachy "Frequency-domain analysis and tuning of the algebraic differentiators" in IJoC., 2018

Implementation

- To implement a continuous-time differentiator, a discretization method is needed.
- Explicit (forward) Euler discretization is mostly utilized to achieve a mere copy of the continuous-time algorithms due to its simplicity (chattering, lack of a proof for the convergence, ...).
- Some studies are dedicated to improve the explicit discretization, e.g., redesigning the parameters, adding high-degree Taylor expansion terms, adding nonlinear terms. However, some drawbacks are inherent.

Arbitrary-order super-twisting differentiator (AO-STD)

Continuous-time AO-STD

$$\begin{cases} \dot{z}_i(t) = -\lambda_i L^{\frac{i+1}{n+1}} \lceil \sigma_0(t) \rfloor^{\frac{n-i}{n+1}} + z_{i+1}(t), \quad i = 0, \dots, n-1 \\ \dot{z}_n(t) \in -\lambda_n L \,\mathrm{sgn}(\sigma_0(t)), \end{cases}$$

$\Downarrow$ Continuous-time system: $\dot{x}(t) = f(x(t))$

Explicit Euler discretization: $x_{k+1} = hf(x_k) + x_k$ $\Downarrow$

Explicit AO-STD

$$\begin{cases} z_{i,k+1} = -h\lambda_i L^{\frac{i+1}{n+1}} \lceil \sigma_{0,k} \rfloor^{\frac{n-i}{n+1}} + hz_{i+1,k} + z_{i,k}, i = 0, \dots, n-1 \\ z_{n,k+1} \in -h\lambda_n L \,\mathrm{sgn}(\sigma_{0,k}) + z_{n,k} \end{cases}$$

z_i: differentiation order i

$\sigma_{0,k} = z_0(t) - f(t)$

n: order of the differentiator

$\lceil a \rfloor^b = |a|^b \,\mathrm{sgn}(a)$

$f(t)$: input

λ_i, L: parameters

h: sampling time

$i = 0, 1, \dots, n-1$

Revisions of the explicit discretization

Explicit AO-STD

$$\begin{cases} z_{i,k+1} = -h\lambda_i L^{\frac{i+1}{n+1}} \lceil \sigma_{0,k} \rfloor^{\frac{n-i}{n+1}} + h z_{i+1,k} + z_{i,k},\, i = 0, \ldots, n-1 \\ z_{n,k+1} \in -h\lambda_n L \operatorname{sgn}(\sigma_{0,k}) + z_{n,k} \end{cases}$$

Explicit HDD

$$\begin{cases} z_{i,k+1} = -h\lambda_i L^{\frac{i+1}{n+1}} \lceil \sigma_{0,k} \rfloor^{\frac{n-i}{n+1}} + \sum_{j=1}^{n-i} \frac{h^j}{j!} z_{j+1,k} + z_{i,k},\, i = 0, \ldots, (n-1) \\ z_{n,k+1} \in -h\lambda_n L \operatorname{sgn}(\sigma_{0,k}) + z_{n,k} \end{cases}$$

Explicit GHDD (third-order)

$$\begin{cases} z_{0,k+1} = z_{0,k} + h z_{1,k} + \frac{h^2}{2} z_{2,k} + \frac{h^3}{6} z_{3,k} + h\psi_{0,k} \\ z_{1,k+1} = z_{1,k} + h z_{2,k} + \frac{h^2}{2} z_{3,k} + h\psi_{1,k} + \alpha_{12} h^2 \psi_{2,k} + \alpha_{13} h^3 \psi_{3,k} \\ z_{2,k+1} = z_{2,k} + h z_{3,k} + h\psi_{2,k} + \alpha_{23} h^2 \psi_{3,k} \\ z_{3,k+1} \in z_{3,k} + h\psi_{3,k}, \end{cases}$$

$\Psi_i(\cdot)$: a discontinuous function

Discretization

Continuous-time system	$\dot{x}(t) = f_1(x(t)) + f_2(x(t))$	X
Explicit discretization	$x_{k+1} = hf_1(x_{\color{red}k}) + hf_2(x_{\color{red}k}) + x_k$	E-X
Implicit discretization	$x_{k+1} = hf_1(x_{\color{blue}k+1}) + hf_2(x_{\color{blue}k+1}) + x_k$	I-X
Semi-implicit discretization	$x_{k+1} = hf_1(x_{\color{blue}k+1}) + hf_2(x_{\color{red}k}) + x_k$ $x_{k+1} = hf_1(x_{\color{red}k}) + hf_2(x_{\color{blue}k+1}) + x_k$	SI-X

- k: explicit variable
- $k+1$: implicit variable

Overview of the sliding-mode-based differentiators

Continuous-time system	Explicit	Implicit	Semi-implicit
STD	E-STD VGED E-STDAC	I-STD	SI-STD
HD-STD	E-HD-STD	I-HD-STD	SI-HD-STD
QD	E-QD	I-QD	-
AO-STD	E-AO-STD HD	I-AO-STD	SI-AO-STD
HDD	E-HDD	I-HDD	-
GHDD	E-GHDD	I-GHDD	-
SHMD	-	I-FDFF I-AO-FDFF	-

- The contributions are indicated in blue.

Discretization of the AO-STD

Continuous-time AO-STD

$$\begin{cases} \dot{z}_i(t) = -\lambda_i L^{\frac{i+1}{n+1}} \lceil \sigma_0(t) \rfloor^{\frac{n-i}{n+1}} + z_{i+1}(t), \quad i = 0, \ldots, n-1 \\ \dot{z}_n(t) \in -\lambda_n L \,\mathrm{sgn}(\sigma_0(t)), \qquad \sigma_0(t) = z_0(t) - f(t) \end{cases} \tag{1}$$

Explicit discretization (E-AO-STD)

$$\begin{cases} z_{i,k+1} = -h\lambda_i L^{\frac{i+1}{n+1}} \lceil \sigma_{0,\color{red}{k}} \rfloor^{\frac{n-i}{n+1}} + hz_{i+1,\color{red}{k}} + z_{i,k} & \text{(2a)} \\ z_{n,k+1} \in -h\lambda_n L \,\mathrm{sgn}(\sigma_{0,\color{red}{k}}) + z_{n,k} & \text{(2b)} \end{cases}$$

Implicit discretization (I-AO-STD)

$$\begin{cases} z_{i,k+1} = -h\lambda_i L^{\frac{i+1}{n+1}} \lceil \sigma_{0,\color{blue}{k+1}} \rfloor^{\frac{n-i}{n+1}} + hz_{i+1,\color{blue}{k+1}} + z_{i,k} & \text{(3a)} \\ z_{n,k+1} \in -h\lambda_n L \,\mathrm{sgn}(\sigma_{0,\color{blue}{k+1}}) + z_{n,k} & \text{(3b)} \end{cases}$$

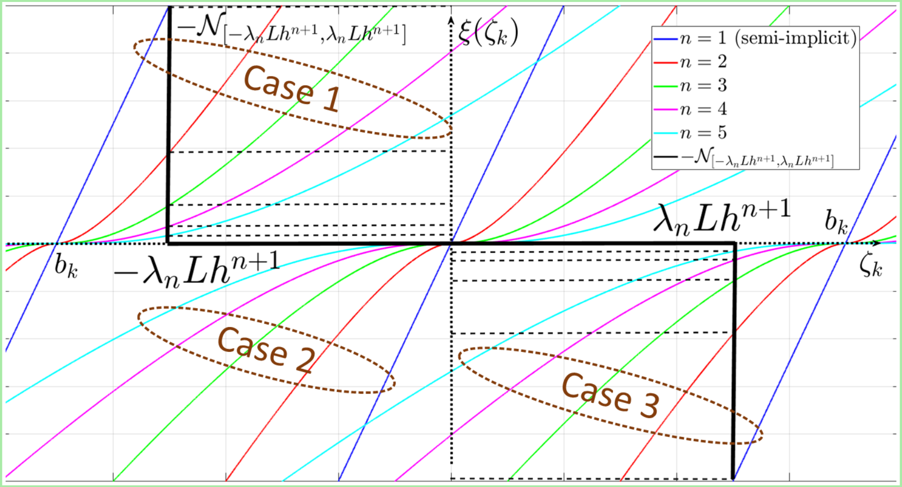
Solving the generalized equation

Generalized equation

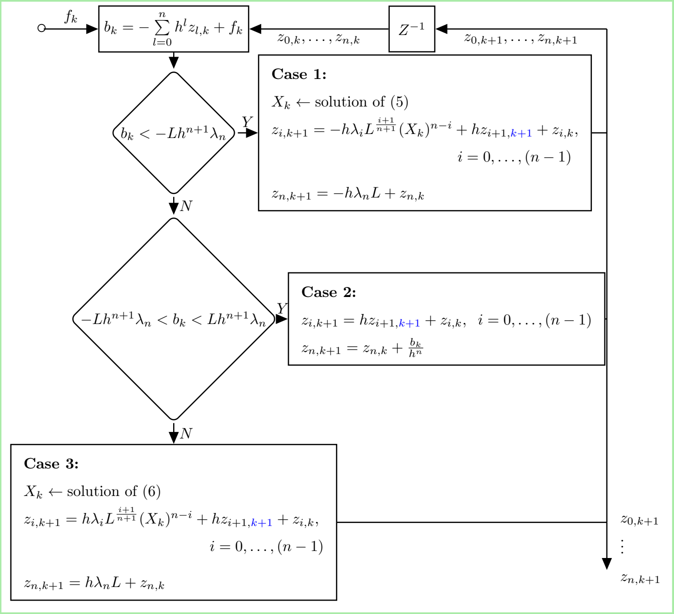
$$
\begin{cases}
g(\sigma_{0,k+1}) \in -h^{n+1}\lambda_n L\,\mathrm{sgn}(\sigma_{0,k+1}) & \text{(4a)}\\
g(\sigma_{0,k+1}) = \sigma_{0,k+1} + \sum_{l=0}^{n-1}\left(h^{l+1}\lambda_l L^{\frac{l+1}{n+1}}\lceil\sigma_{0,k+1}\rfloor^{\frac{n-l}{n+1}}\right) + b_k & \text{(4b)}\\
b_k = -\sum_{l=0}^{n} h^l z_{l,k} + f_k\ , \qquad \xi(\sigma_{0,k+1}) = g^{-1}(\sigma_{0,k+1}). & \text{(4c)}
\end{cases}
$$

Solving the generalized equation

Graphical interpretation of the generalized equation



Flowchart of the I-AO-STD



Implementation of the full-implicit schemes

$b_k < -h^{n+1}\lambda_n L \quad \rightarrow \quad x_k \triangleq \sigma_{0,k+1}^{\frac{1}{n+1}}$

$$x_k^{n+1} + \sum_{l=0}^{n-1} \left(h^{l+1}\lambda_l L^{\frac{l+1}{n+1}} x_k^{n-l}\right) + b_k + h^{n+1}\lambda_n L = 0 \tag{5}$$

$b_k > h^{n+1}\lambda_n L \quad \rightarrow \quad x_k \triangleq -\sigma_{0,k+1}^{\frac{1}{n+1}}$

$$-x^{n+1} - \sum_{l=0}^{n-1} \left(h^{l+1}\lambda_l L^{\frac{l+1}{n+1}} x^{n-l}\right) + b_k - h^{n+1}\lambda_n L = 0 \tag{6}$$

Different semi-implicit schemes for the AO-STD

Full implicit discretization (I-AO-STD)

$$\begin{cases} z_{0,k+1} = -h\lambda_0 L^{\frac{1}{2}} \lceil \sigma_{0,k+1} \rfloor^{\frac{1}{2}} + hz_{1,k+1} + z_{0,k} & \text{(7a)} \\ z_{1,k+1} \in -h\lambda_1 L \operatorname{sgn}(\sigma_{0,k+1}) + z_{1,k} & \text{(7b)} \end{cases}$$

A semi-implicit scheme (SI-AO-STD)

$$\begin{cases} z_{0,k+1} = -h\lambda_0 L^{\frac{1}{2}} \lceil \sigma_{0,k} \rfloor^{\frac{1}{2}} + hz_{1,k+1} + z_{0,k} & \text{(8a)} \\ z_{1,k+1} \in -h\lambda_1 L \operatorname{sgn}(\sigma_{0,k+1}) + z_{1,k} & \text{(8b)} \end{cases}$$

Another semi-implicit scheme (SI-AO-STD)

$$\begin{cases} z_{0,k+1} = -h\lambda_0 L^{\frac{1}{2}} \lceil \sigma_{0,k} \rfloor^{\frac{1}{2}} + hz_{1,k+1} + z_{0,k} & \text{(9a)} \\ z_{1,k+1} \in -h\lambda_1 L \operatorname{sgn}(\sigma_{0,k}) + z_{1,k} & \text{(9b)} \end{cases}$$

Parameter selection for the I-AO-STD

$$\begin{cases} \underbrace{\left|\sum_{i=1}^{n-1}\left((n-i)h^i f_{0,k}^{(i)}\right) + f_{0,k+1} - f_{0,k}\right| < Lh^{n+1}\lambda_n}_{\text{exactness on the base signal } (f_{0,k})} & \text{(10a)} \\ \underbrace{\left|\sum_{i=1}^{n-1}\left((n-i)h^i n_k^{(i)}\right) + n_{k+1} - n_k\right| \gg Lh^{n+1}\lambda_n}_{\text{cancellation of the exactness on the noise } (n_k)} & \text{(10b)} \end{cases}$$

$f_K = f_{0,K} + n_K$: input	$f_{0,K}$: base signal	n_k: noise
n: order of the differentiator	i: order of the output	
h: sampling time	L, λ_n: parameters	

Fundamental operators

I-STD: $\sigma_{0,k+1} \mapsto \left(I_d + a\lceil\cdot\rfloor^{\frac{1}{2}} + L\lambda_1 h^2\,\mathrm{sgn}(\cdot)\right)^{-1}(-b_k)$

I-HD-STD: $\sigma_{0,k+1} \mapsto \left(I_d + h\lambda_0 L^{\frac{1}{2}}\left(\lceil\cdot\rfloor^{\frac{1}{2}} + \mu\lceil\cdot\rfloor^{\frac{3}{2}}\right) + h^2\lambda_1 L\left(\frac{1}{2}\,\mathrm{sgn}(\cdot) + 2\mu(\cdot) + \frac{3}{2}\mu^2\lceil\cdot\rfloor^2\right)\right)^{-1}(-b_k)$

I-AO-STD: $\sigma_{0,k+1} \mapsto \left(I_d + \sum_{l=0}^{n-1}\left(h^{l+1}\lambda_l L^{\frac{l+1}{n+1}}\lceil\cdot\rfloor^{\frac{n-l}{n+1}}\right) + h^{n+1}\lambda_n L\,\mathrm{sgn}(\cdot)\right)^{-1}(-b_k)$

I-HDD: $\sigma_{0,k+1} \mapsto \left(I_d + \sum_{l=0}^{n-1}\left(m_l h^{l+1}\lambda_l L^{\frac{l+1}{n+1}}\lceil\cdot\rfloor^{\frac{n-l}{n+1}}\right) + h^{n+1}\lambda_n L m_n\,\mathrm{sgn}(\cdot)\right)^{-1}(-b_k)$

I-GHDD: $\sigma_{0,k+1} \mapsto \left(I_d - \sum_{i=0}^{n-1}\left(h^{i+1}\psi_{i,k+1}(\cdot)\right) + h^{n+1}L\lambda_n\,\mathrm{sgn}(\cdot)\right)^{-1}(-b_k)$

I-FDFF:

I-AO-FDFF: $\sigma_{0,k+1} \mapsto \left(aI_d + c\,\mathrm{sgn}(\cdot)\right)^{-1}(-b_k)$

SI-STD: $\sigma_{0,k+1} \mapsto \left(I_d + h^2\lambda_1 L\,\mathrm{sgn}(\cdot)\right)^{-1}(-b_k)$

SI-AO-STD: $\sigma_{0,k+1} \mapsto \left(I_d + a\lceil\cdot\rfloor^{\frac{1}{2}} + L\lambda_1 h^2\,\mathrm{sgn}(\cdot)\right)^{-1}(-b_k)$

General form of the fundamental operator

$$\sigma_{0,k+1} \mapsto \big(I_d + (\cdot)\,\mathrm{sgn}(\cdot)\big)^{-1}(-b_k)$$

where

- $\sigma_{0,k+1}$: the implicit variable
- I_d: identity function, i.e., $x \mapsto x$
- $(\cdot)^{-1}$: inverse of mapping, possibly set-valued
- b_k: a function of current states, i.e., $\sigma_{i,k}\ , \quad i = 0, \ldots, n$

Analytical results related to the I-AO-STD

1. AO-STD has a Lyapunov function with convex level set (n=1).
2. Sliding-surface of the I-AO-STD is invariant.
3. Conditions for the exactness are derived.
4. I-AO-STD is insensitive to gains during the sliding-phase.
5. I-AO-STD eliminates the chartering inherently.
6. Asymptotic stability of the I-AO-STD is ensured.
7. Finite-time convergence of I-AO-STD is studied ($n+1$ steps are required).
8. Well-posedness of the I-AO-STD is addressed.
9. Investigation of the Levant's inequality.

Structure of the simulations

1. Levant's inequality
2. Validating some theorems
3. Noise-free case
4. White noise
5. Sinusoidal noise
6. Bell-shaped noise
7. Quantization
8. Transient responses
9. Higher-order differentiation
10. Effect of the solver
11. Effect of the criteria on the optimization
12. Effect of the sampling time

Investigation of the Levant's inequality

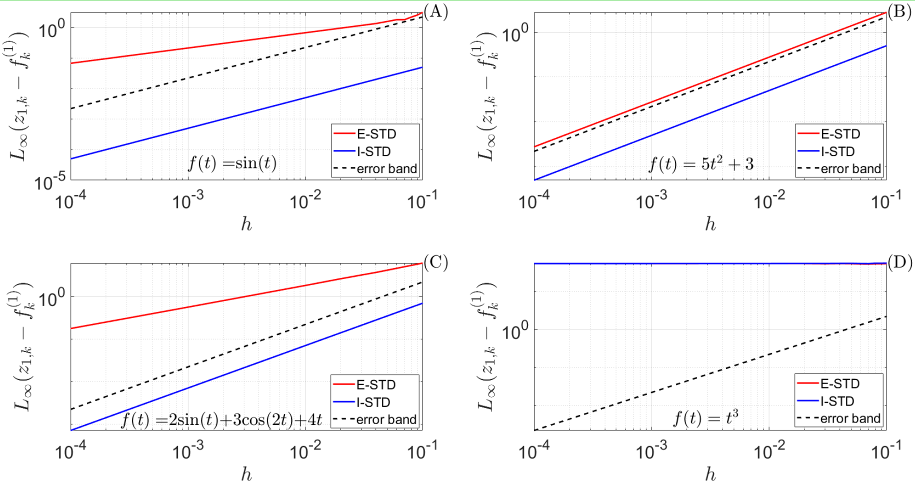
Levant's inequality

$$|z_{i,k} - f_k^{(i)}| < \mu_i h^{n-i+1}$$

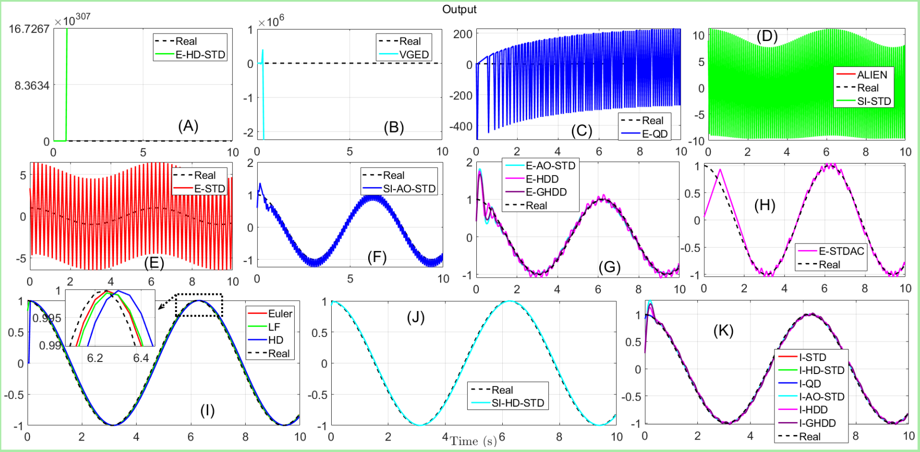
- $f_k^{(i)}$: differentiation order i of f_k
- z_i: estimation of $f_k^{(i)}$
- μ_i: a constant
- i: the differentiation-order of the output $(i = 0, \ldots, n-1)$
- n: the order of the differentiator

Investigation of the Levant's inequality

error band: $|z_{i,k} - f_k^{(i)}| = \mu_i h^{n-i+1}$ $\mu_i = L\lambda_n$



Arbitrary over-sized gains   input: sin(t)   h=50ms



Objective functions

- $\bar{L}_2(e_k) = \frac{h}{t_f}||e_k|| = \frac{h}{t_f}\sqrt{\sum_{k=0}^{t_f/h} e_k^2}$
- $L_\infty(e_k) = ||e_k||_\infty = \max_k |e_k|, \quad k = 0, \ldots, t_f/h$
- $\text{VAR}(y_k) = \sum_{k=0}^{t_f/h} |y_k - y_{k-1}|$
- $\text{THD}(y_k) = 100\frac{\sqrt{\sum_k V_k^2}}{V_0}, \quad k = 0, \ldots, t_f/h,$

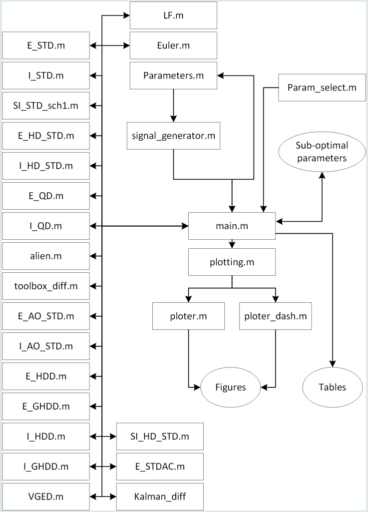
- y_k: output
- e_k: error
- v_k: frequency component
- t_f: final time
- h: sampling time

Toolbox overview

Features

- 24 different methods and their variants (cascade setups, ...)
- Higher-order differentiations (up to order 8)
- Built-in tuning algorithm
- Realistic conditions
- Several types of plots and performance functions
- Comparative analysis, and validating the theorems
- Simulink blocks
- Generating results in LaTeX

Diagram



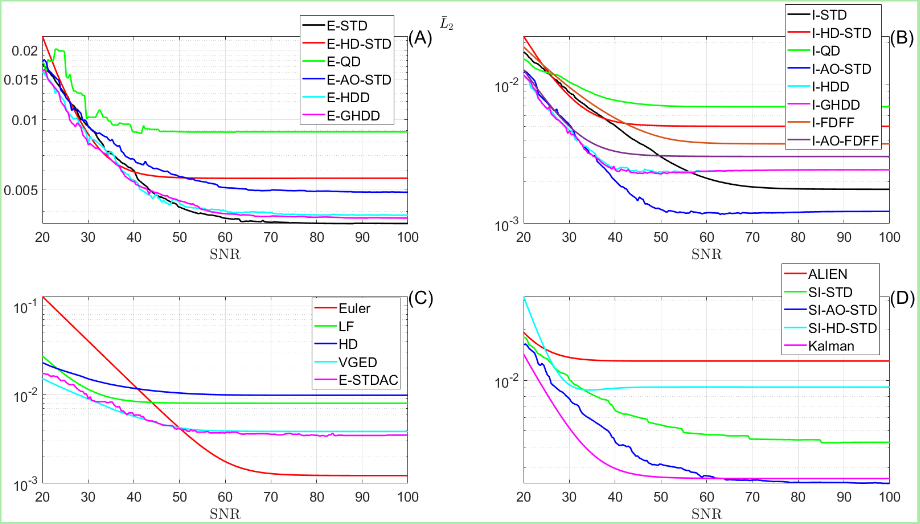
$h = 50$ms, SNR=30dB, $f(t) = \sin(t) + n(t)$

Method	Parameters	$J = 10000\bar{L}_2(e_k)$
Euler	No parameter	400.7426
LF	c=7.1113	114.5675
E-STD	L=0.7713	92.7441
I-STD	L=0.7324	87.1980
SI-STD	L=0.6985	101.2067
E-HD-STD	L=0.0770, μ=20.8386	86.7613
I-HD-STD	L=0.1021, μ=21.2075	82.9217
SI-HD-STD	L=0.1434, μ=93.5748	94.3047
E-QD	F=4.4026, α= 0.3780	102.6753
I-QD	F=4.5323, α=0.8123	104.6224
ALIEN	T=0.5020 , κ=1 , μ=2	137.2458
HD	r=2.5655	150.2620
E-AO-STD	L=4.8973	93.1914
I-AO-STD	L=2.9122	47.9806
SI-AO-STD	L=2.8157	75.5441
E-HDD	L=4.9392	79.3572
E-GHDD	L=4.8970	77.8480
I-HDD	L=2.9921	44.3107
I-GHDD	L=2.9822	43.4911
VGED	μ=4.3694, τ=1.3269, ω_c=12.2205, q=0.2997	89.2798
E-STDAC	α=0.5318, ϵ=0.0000	89.5387
I-FDFF	ω_s=19.6607, ω_f=8.4727, ρ=8.6929, γ=0.0348	95.9795
I-AO-FDFF	F=37.7845, ϵ=18.6061, ω_s=2.5068 ω_f=62.6396, α_1=456.7015, ρ=88.3003	50.2447
Kalman	$R = 8.4121 \times 10^{-4}$	51.9665

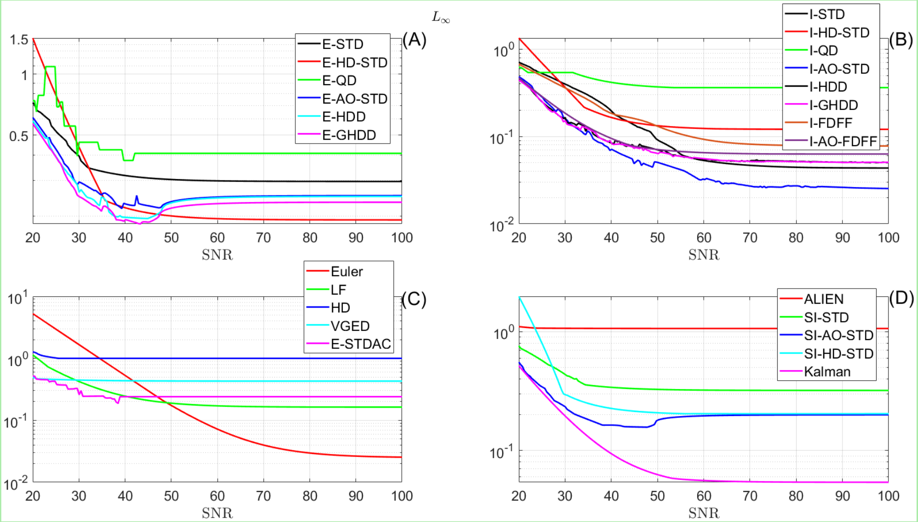
$h = 50\text{ms}$, SNR=30dB, $f(t) = \sin(t) + n(t)$

Method	$\bar{L}_2$	$\tilde{L}_2$	L_∞	VAR	THD%	Calculation time
Euler	0.0401	0.6328	1.6678	156.1079	10.8251	1.00 β
LF	0.0115	0.2312	0.4237	27.2973	5.1186	1.39 β
E-STD	0.0093	0.1813	0.3950	22.6491	5.0380	1.75 β
I-STD	0.0087	0.1694	0.3991	22.3047	4.8264	1.79 β
SI-STD	0.0101	0.1971	0.4364	22.6797	5.0543	1.60 β
E-HD-STD	0.0087	0.1733	0.4380	21.2378	4.9033	1.59 β
I-HD-STD	0.0083	0.1647	0.3669	21.3538	4.7479	28.40 β
SI-HD-STD	0.0094	0.2021	0.2948	12.6772	4.9159	1.98 β
E-QD	0.0103	0.2137	0.4616	23.8976	4.8946	1.92 β
I-QD	0.0105	0.2165	0.5410	23.6339	4.7218	2.00 β
ALIEN	0.0137	0.2937	1.0670	9.8502	3.4701	13.64 β
HD	0.0150	0.3140	0.9988	26.3416	4.3390	7.37 β
E-AO-STD	0.0093	0.2025	0.2947	11.7065	4.7248	2.60 β
I-AO-STD	0.0048	0.1032	0.1565	8.6579	4.4372	27.27 β
SI-AO-STD	0.0076	0.1651	0.2333	10.3001	4.6059	3.65 β
E-HDD	0.0079	0.1707	0.2623	11.7419	4.6817	3.45 β
E-GHDD	0.0078	0.1682	0.2496	11.0980	4.6572	4.44 β
I-HDD	0.0044	0.0948	0.1454	8.7591	4.4111	27.19 β
I-GHDD	0.0043	0.0935	0.1420	8.4464	4.4002	24.47 β
VGED	0.0089	0.1889	0.4458	16.1570	5.0126	12.59 β
E-STDAC	0.0090	0.1976	0.2581	11.8332	4.3820	2.38 β
I-FDFF	0.0096	0.1975	0.3608	23.3846	4.9785	1.77 β
I-AO-FDFF	0.0050	0.1069	0.1853	10.4184	4.4473	11.15 β
Kalman	0.0052	0.1125	0.1952	8.4625	4.3418	10.09 β

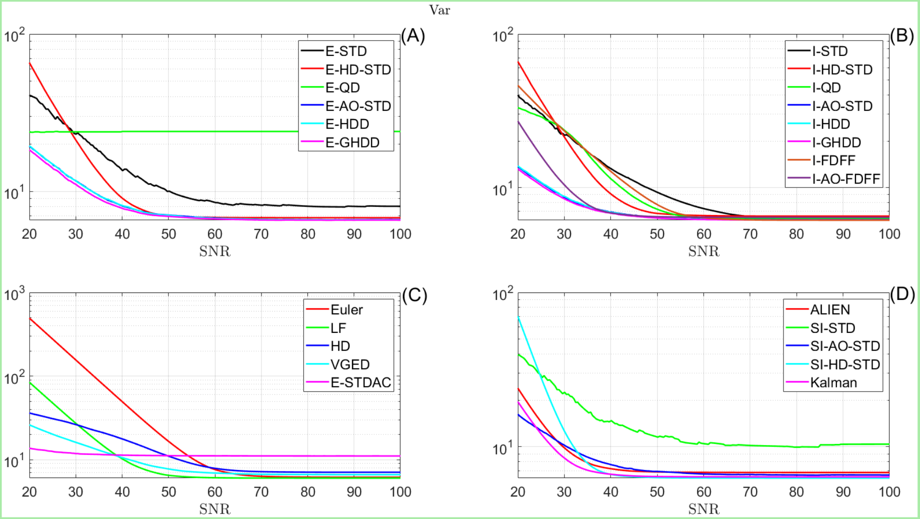
White noise, $f(t) = \sin(t) + n(t)$, $h = 50$ms



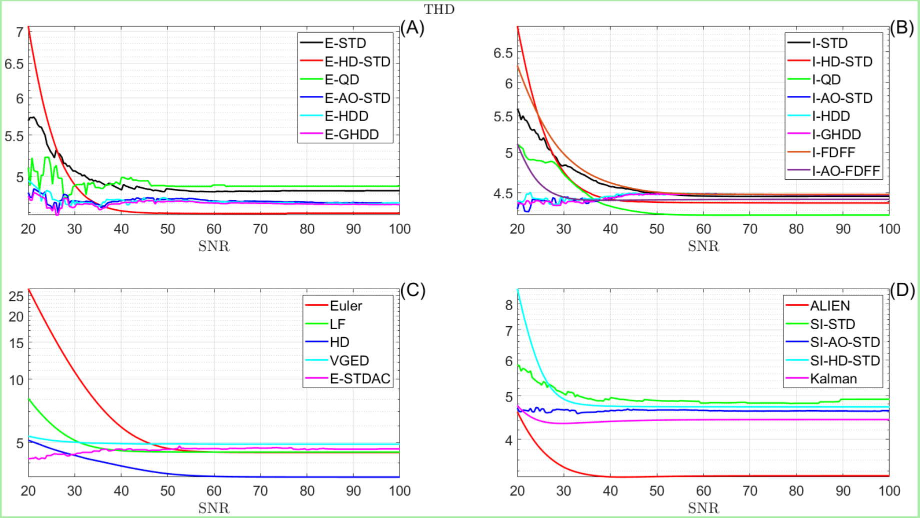
White noise, $f(t) = \sin(t) + n(t)$, $h = 50\text{ms}$



White noise, $f(t) = \sin(t) + n(t)$, $h = 50\text{ms}$



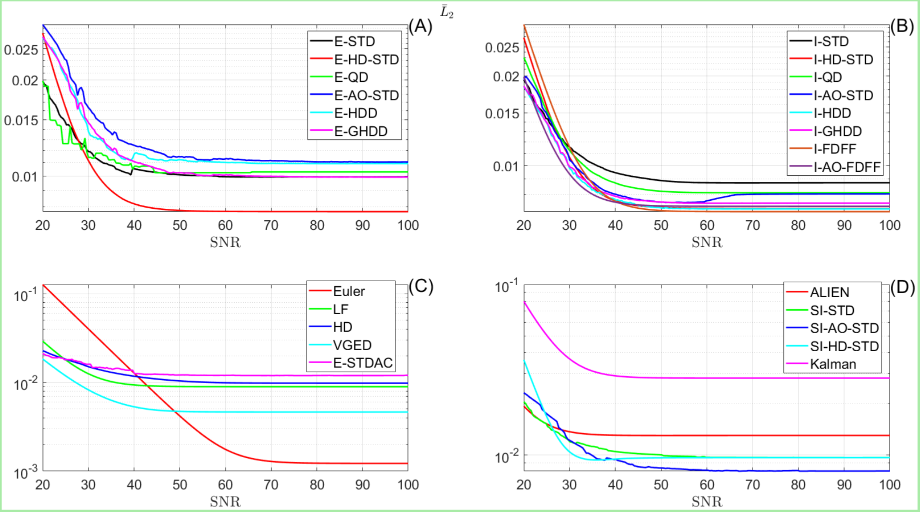
White noise, $f(t) = \sin(t) + n(t)$, $h = 50\text{ms}$



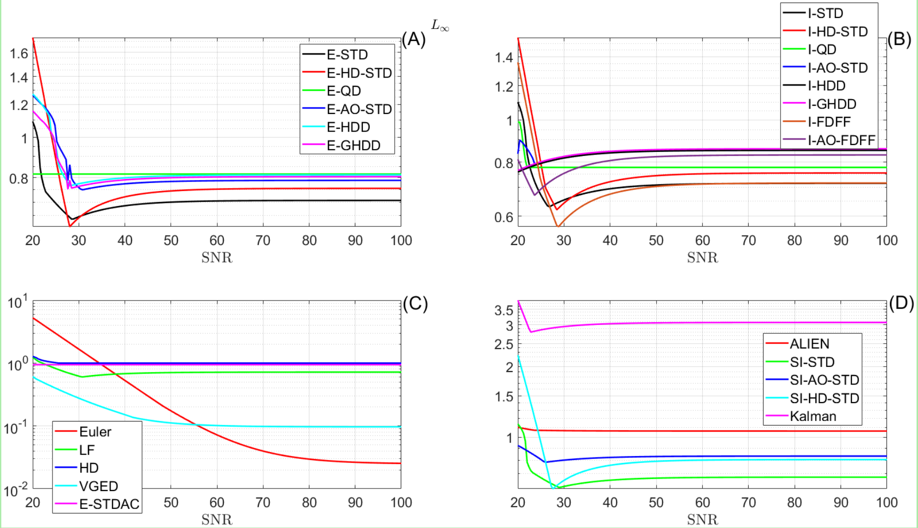
+initial error, $h = 50$ms, SNR=30dB, $f(t) = \sin(t) + n(t)$

Method	Parameters	$J = 10000\bar{L}_2(e_k)$
Euler	No parameter	400.7426
LF	c=7.7652	125.2666
E-STD	L=0.7621	119.3866
I-STD	L=0.7375	114.6544
SI-STD	L=0.7105	120.0440
E-HD-STD	L=0.1526, μ=17.2523	111.7988
I-HD-STD	L=0.1842, μ=19.2301	106.3642
SI-HD-STD	L=0.1779, μ=96.4825	104.6239
E-QD	F=3.6398, α= 0.5345	114.1485
I-QD	F=4.4258, α=108.0366	112.2623
ALIEN	T=0.5020 , κ=1 , μ=2	137.2458
HD	r=2.5653	150.2620
E-AO-STD	L=20.4049	160.8194
I-AO-STD	L=14.3801	102.5989
SI-AO-STD	L=9.7812	120.3473
E-HDD	L=15.9819	137.7374
E-GHDD	L=20.4050	144.8012
I-HDD	L=14.3671	96.9191
I-GHDD	L=15.7542	98.9933
VGED	μ=6.3813, τ=5.9048, ω_c=12.1897, q=0.0011	82.4729
E-STDAC	α=0.7896, ϵ=0.0018	160.3342
I-FDFF	ω_s=59.4160, ω_f=22.6361, ρ=12.6443, γ=0.0001	115.2277
I-AO-FDFF	F=53.7596, ϵ=22.9829, ω_s=3.7986 ω_f=37.5615, α_1=54.2768, ρ=29.4002	93.5342
Kalman	$R = 3.5920 \times 10^{-8}$	366.6154

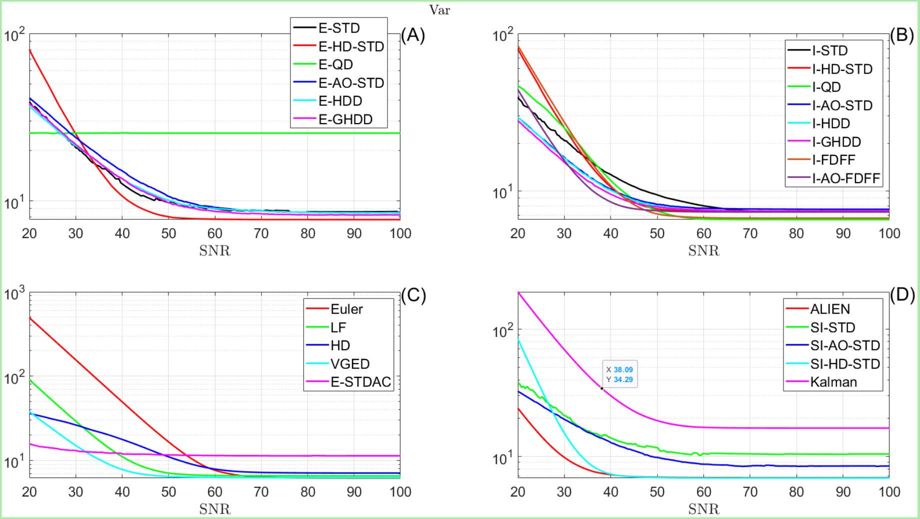
+initial error, $h = 50\text{ms}$, $f(t) = \sin(t) + n(t)$



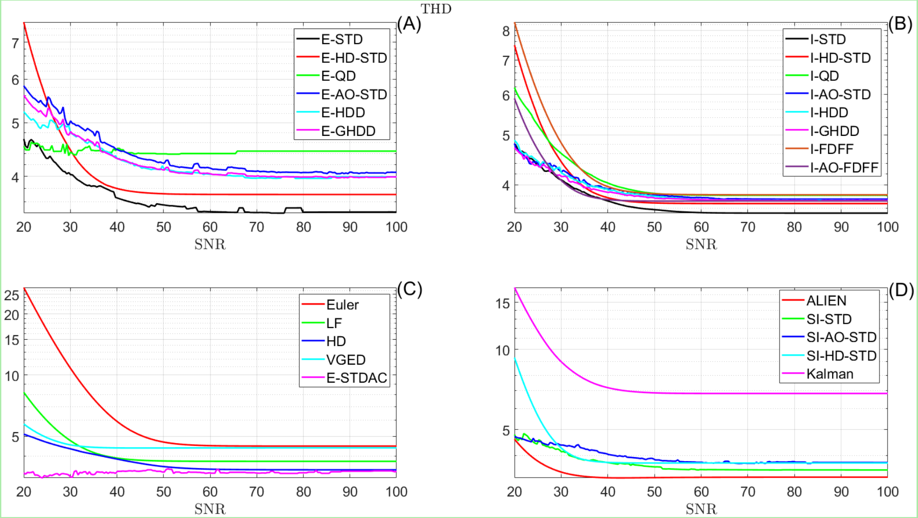
+initial error, $h = 50\text{ms}$, $f(t) = \sin(t) + n(t)$



+initial error, $h = 50\text{ms}$, $f(t) = \sin(t) + n(t)$



+initial error, $h = 50\text{ms}$, $f(t) = \sin(t) + n(t)$



Summarized results

Method	noise-free	white noise	sinusoidal noise	bell-shaped noise	quantization
Euler	2 ∞ V C	2 ∞ V T C	2 ∞ V T C	2 ∞ V T C	2 ∞ V T C
LF	V C	C	C	C	C
E-STD	C	C	C	C	C
I-STD	2 V C	C	-	-	C
SI-STD	C	C	C	C	C
E-HD-STD	C	C	-	-	C
I-HD-STD	C	C	C	C	C
SI-HD-STD	V C	C	-	-	C
E-QD	V C	C	-	-	C
I-QD	V	-	-	-	-
ALIEN	2 ∞ T C	∞ T C	∞ T C	∞ T C	∞ V T C
HD	2 ∞ T C	∞	∞ T C	∞ T	∞ C
E-AO-STD	-	-	-	-	-
I-AO-STD	2 ∞ V C	2 ∞ V C	2 ∞ V C	2 ∞ V C	2 ∞ V C
SI-AO-STD	-	-	-	-	-
E-HDD	-	-	-	-	-
E-GHDD	-	-	-	-	-
I-HDD	V C	2 ∞ V C	2 ∞ V C	2 ∞ V C	2 ∞ V C
I-GHDD	V C	2 ∞ V C	2 ∞ C	2 ∞ V C	2 ∞ V C
VGED	C	-	-	-	-
E-STDAC	C	-	V	-	C
I-FDFF	V	C	-	-	C
I-AO-FDFF	C	2 ∞ C	2 ∞ V C	C	2 ∞ C
Kalman*	-	2 ∞ V	2 ∞ V T	V	2 ∞ V

2 : $\bar{L}_2(e)$, ∞ : $L_\infty(e)$, V: var, T: THD, C: calculation time

blue= best , red: worst, *the worst transient response

Main results

- Explicit discretization should be avoided.
- Implicit differentiators supersede the linear filters.
- Generally, I-AO-STD, I-HDD, and I-GHDD present the best responses.
- Increasing the order of a differentiator generally improves the robustness to noise. However, it increases the transient time.
- Kalman presents one of the worst transient responses.
- Semi-implicit schemes can be utilized in applications with limited resources to provide a compromise between the performance and the calculation time.
- Newton and Halley's algorithms are suitable iterative schemes to solve the generalized equations for implicit methods.

Future works

- Providing strict Lyapunov functions with convex level sets for the AO-STD ($n > 1$)
- Levant's inequality for the I-AO-STD ($n > 1$)
- Investigation of the differentiators in the closed-loop systems
- Practical experiments
- Using homogeneity theorem to study the exact differentiators
- Developing more efficient solvers
- Optimizing the structure of the differentiators (Exact-ALIEN)
- Addressing the parameter design more clearly (addressing filtration)

Discussion (open questions)

1. Possible case studies and laboratory set-ups
2. How to tune the parameters in practical closed-loop systems?
3. Objective functions for tuning the parameters in closed-loop systems (estimation error, output tracking error, ...)
4. How to identify the measurement noise corresponds to the real laboratory set-ups (for the simulations and parameter tuning)?
5. Would we need extra filtration stages in closed-loops?

Experimental Results of Controllers and Differentiators/Observers

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Sliding Mode Control (SMC)

Evolution:

- *Classical SMC* - contributed by Prof. Utkin [Itkis, 1976], [Utkin, 1977], [Utkin, 1992].
- *HOSM* - Prof. Levant [Levant and Levantovsky, 1993]
- Combination of the *Classical* and HOSM [Shtessel et al., 2014]
- *SMC* in *discrete-time* [Kukrer and Makhamreh, 2018], [Sira-Ramirez, 1991]
- *Explicit and Implicit SMC* [Galias and Yu, 2006], [Galias and Yu, 2009], [Acary and Brogliato, 2010a], [Acary et al., 2012]

Objectives:

- Robustness against uncertainties and perturbations
- Finite-time convergence

Time-Discretization

Taking the general representation of a non-linear system (in differential equation):

$$\dot{x} = f(x, t)$$

It could be discretized (in difference equation) as:

$$x^+ = F(x, x^+, t, t^+)$$

Example:

$$\dot{x}(t) = -Ku \quad \text{(Continuous)} \qquad x^+ = x - Khu \quad \text{(Discrete)}$$

Concept

Interesting methods for performance comparison ↑↑

Explicit

$$\begin{cases} x^+ = x - Khu, \\ u = \operatorname{sgn}(x) \end{cases}$$

Implicit

$$\begin{cases} x^+ = x - Khu^+, \\ u^+ = \operatorname{sgn}(x^+) \end{cases}$$

Chattering: **YES** **NO**

[Galias and Yu, 2006], [Galias and Yu, 2007]

Remarks:

- $|x| \geq Kh$ – no difference
- $0 < x < Kh$ – chattering introduced in explicit method

Concept of Implicit projector

Challenge: How $\text{sgn}(x^+)$ could be used in x^+? Seems to create an algebraic loop error in implementation...!

Idea: To design a projector

- Inversion of set-valued mapping:

$$u^+ = \text{sgn}(x^+) \Longleftrightarrow x^+ \in \mathcal{N}_{[-1,1]}(u^+)$$

- Mathematical representations:

Sign function **Normal-cone**

$$\text{sgn}(x) = \begin{cases} -1, & \text{if } \; x \in \mathbb{R}^- \\ 1, & \text{if } \; x \in \mathbb{R}^+ \\ 0, & \text{if } \; x = 0 \end{cases} \qquad \mathcal{N}_{[-1,1]}(u^+) = \begin{cases} \mathbb{R}^-, & \text{if } \; u^+ = -1 \\ \mathbb{R}^+, & \text{if } \; u^+ = 1 \\ 0, & \text{if } \; |u^+| < 1 \end{cases}$$

Homogeneous Differentiators

↑ **Explicit Levant's second-order homogeneous**

$$\begin{cases} z_1^+ = z_1 + h(\lambda_1 \lceil e_1 \rfloor^{\alpha} + z_2) \\ z_2^+ = z_2 + h\lambda_2 \lceil e_1 \rfloor^{2\alpha-1} \end{cases}$$

where $\lceil e_1 \rfloor^{\gamma} = |e_1|^{\gamma} \operatorname{sgn}(e_1)$ and $e_1 = y - z_1$.

[1] *Implicit discretization concept*

Implicit:

$$\begin{cases} z_1^+ = z_1 + h\left(\lambda_1 \tilde{u} + z_2\right) \\ z_2^+ = z_2 + h\lambda_2 \tilde{u} \\ \tilde{u} = \mathcal{N}(e_1, \lambda_1) \end{cases}$$

Semi-Implicit

$$\begin{cases} z_1^+ = z_1 + h\left(\lambda_1 |e_1|^{\alpha} \tilde{u} + z_2\right) \\ z_2^+ = z_2 + h\lambda_2 |e_1|^{2\alpha-1} \tilde{u} \\ \tilde{u} = \mathcal{N}(e_1, \alpha_1, \lambda_1) \end{cases}$$

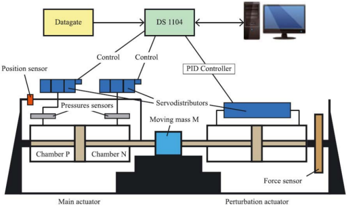
[1] [Acary and Brogliato, 2010b], [Brogliato and Polyakov, 2015], [Michel et al., 2020]

Objectives

Primary Points:

- Formulation of controllers and differentiators based on explicit and implicit methods.
- Tests on simple systems for conceptual understanding.
- Tests on the simulator model.
- Implementation on the real-time system.
- Performance analysis and conclusions.

Set-up introduction



EPA setup (Left) and Control scheme ([Girin and Plestan, 2009]).

- *Desired* – actuator position control and state estimations.
- *Challenge* – to try to suppress or overcome the perturbation effects.
- *Only available measure* – Piston's position y_m.

System Dynamics

A simplified system model, under few assumptions [Girin and Plestan, 2009], is defined as follows:

$$\dot{p}_P = \frac{krT}{V_P(y)}\left[\varphi_P(p_p) + \psi_P(p_P, \text{sgn}(u))u - \frac{S}{rT}p_P v\right]$$

$$\dot{p}_N = \frac{krT}{V_N(y)}\left[\varphi_N(p_N) - \psi_N(p_N, \text{sgn}(-u))u + \frac{S}{rT}p_N v\right]$$

$$\dot{v} = \frac{1}{M}[S(p_P - p_N) - b_v v - F_{ext}]$$

$$\dot{y} = v$$

Control methods

From. . .

Explicit

$$\begin{cases} x^+ = x - Khu, \\ u = \text{sgn}(x) \end{cases}$$

Implicit

$$\begin{cases} x^+ = x - Khu^+, \\ u^+ = \text{sgn}(x^+) \end{cases}$$

With $K = 1$, the following comparisons are made:

- with different initial conditions
- with different sampling periods

Summary

Summary on the convergence ($x \to 0$)

x_0	h	**Explicit Control**	**Implicit-Euler Control**
20	10	Slow Con of x, no B-B in u^+	Fast Con of x, no B-B in u^+
0.2	10	Ch in x, B-B in u	No Ch in x, no B-B in u^+
0.7	10	Ch in x, B-B in u	No Ch in x, no B-B in u^+
20	1	Con of x, u with no Ch	No Ch in x, no B-B in u^+
20	0.1	No Ch in x, B-B in u	No Ch in x, no B-B in u^+
20	0.01	No Ch in x, B-B in u	No Ch in x, no B-B in u^+
20	21	Ch in x, B-B in u	No Ch in x, no B-B in u^+
20	25	Ch in x, B-B in u	No Ch in x, no B-B in u^+
20	50	Ch in x, B-B in u	No Ch in x, no B-B in u^+
20	100	Ch in x, B-B in u	No Ch in x, no B-B in u^+

Con – convergence, Ch – chattering, B-B – bang-bang

With $x_0 = 20$ and $h = 10$,

- **Increasing gain** $K \to$ increase in chattering, no convergence.
- **Decreasing gain** $K \to$ increase in convergence time, no chattering.

Specification concerned

- **Simplified system:**

$$\begin{cases} \dot{y} = v \\ \dot{v} = \frac{1}{M}\left[S(p_P - p_N) - b_v v - F_{ext}\right] \end{cases}$$

(assuming $M = 3.4$ kg and $b_v = 50$)

- **References:**

$$\begin{cases} y_{ref} = A\sin(2\pi f t), \\ \dot{y}_{ref} = 2A\pi f\cos(2\pi f t), \\ \ddot{y}_{ref} = -4A\pi^2 f^2\sin(2\pi f t), \end{cases}$$

(taking $A = 0.04$ (i.e., 40 mm) and $f = 0.1$ Hz).

Second-order Implicit Projector-based Control Law:

- **Synthesis:** Defining sliding surfaces:

$$\begin{cases} \sigma = k_1(y_{ref} - y) + (\dot{y}_{ref} - v) \\ \dot{\sigma} = k_2(\dot{y}_{ref} - v) + (\ddot{y}_{ref} - a) \end{cases}$$

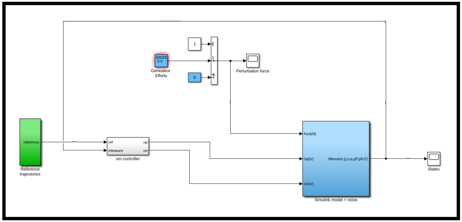
$\{k_1, k_2\} \rightarrow$ controller gains

- **Control input:** New control law given by:

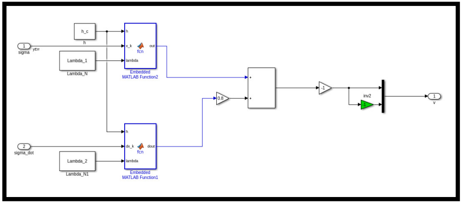
$$w = -Ku = -K(\mathcal{N}(\sigma) + \beta\mathcal{N}(\dot{\sigma}))$$

- $\mathcal{N}(\sigma), \mathcal{N}(\dot{\sigma})$ replacing $\text{sgn}(\sigma), \text{sgn}(\dot{\sigma})$

Implementation



Simulator

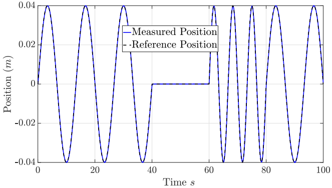


Controller

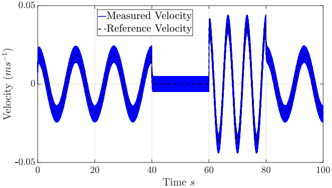
Parameters:

- $k_1 = k_2 = 80$
- $K = 1$
- $\lambda_1 = 0.3$
- $\lambda_2 = 2$
- $h = 0.2$ ms

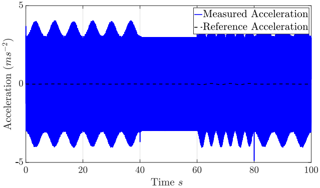
Results



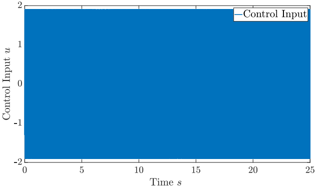
Position



Velocity



Acceleration



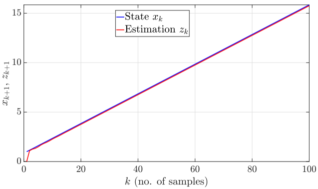
Control input u

Remarks

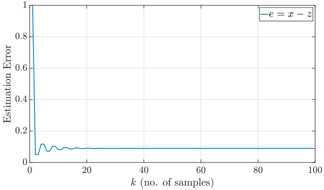
- Good control achieved on the actuator position.
- Chattering observed in the actuator velocity – to be improved (or tuned).
- No effective control on the actuator acceleration – could, for instance, require a *third-order* control law.

First-order autonomous system (Pure Implicit)

- **System:** $x^+ = x + hP$
- **Observer:** $z^+ = z + h\tilde{u}$
- **Correction term:** $\tilde{u} = \begin{cases} \frac{e}{hP}, & |e| < hP \\ \mathrm{sgn}(e), & \text{elsewhere} \end{cases}$



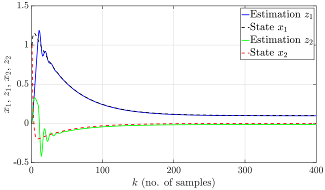
Estimation



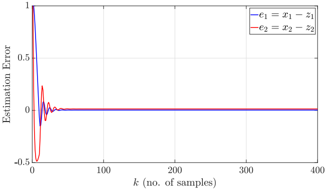
Error

Second-order autonomous system (Pure Implicit)

- **System:** $\begin{cases} x_1^+ = x_1 + hx_2 \\ x_2^+ = x_2 - hx_1 - 5hx_2 + hP \end{cases}$
- **Observer:** $\begin{cases} z_1^+ = z_1 + hz_2 + h\lambda_1\tilde{u} \\ z_2^+ = z_2 - hz_1 - 5hz_2 + h\lambda_2\tilde{u} \end{cases}$
- **Correction term:** $\tilde{u} = \begin{cases} \frac{e}{h\lambda}, & |e| < h\lambda \\ \text{sgn}(e), & \text{elsewhere} \end{cases}$



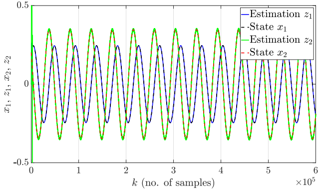
Estimations



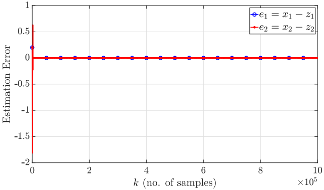
Errors

Second-order autonomous system (Semi-Implicit)

- **System:** $\begin{cases} x_1^+ = x_1 + hx_2 \\ x_2^+ = x_2 - 2hx_1 + hP \end{cases}$
- **Differentiator:** $\begin{cases} z_1^+ = z_1 + hz_2 + h\lambda_1\mu|e_1|^{\alpha}\tilde{u} \\ z_2^+ = z_2 + h\lambda_2\mu^2|e_1|^{2\alpha-1}\tilde{u} \end{cases}$
- **Correction term:** $\tilde{u} = \begin{cases} \frac{\lceil e_1 \rfloor^{1-\alpha}}{h\lambda_1}, & |e_1|^{1-\alpha} < h\lambda_1 \\ \text{sgn}(e_1), & \text{elsewhere} \end{cases}$



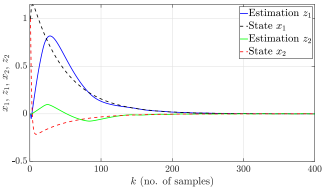
Estimations



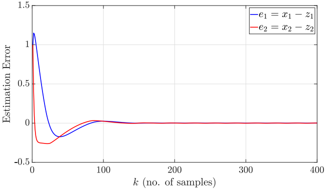
Errors

Second-order autonomous system (Semi-Implicit)

- **System:** $\begin{cases} x_1^+ = x_1 + hx_2 \\ x_2^+ = x_2 - hx_1 - 5hx_2 \end{cases}$
- **Differentiator:** $\begin{cases} z_1^+ = z_1 + hz_2 + h\lambda_1\mu|e_1|^{\alpha}\tilde{u}, \\ z_2^+ = z_2 + h\lambda_2\mu^2|e_1|^{2\alpha-1}\tilde{u}, \end{cases}$
- **Correction term:** $\tilde{u} = \begin{cases} \frac{\lceil e_1 \rfloor^{1-\alpha}}{h\lambda_1}, & |e_1|^{1-\alpha} < h\lambda_1 \\ \mathrm{sgn}(e_1), & \text{elsewhere} \end{cases}$



Estimations



Errors

Variable exponent differentiators

- **Synthesis:** [Ghanes et al., 2017]

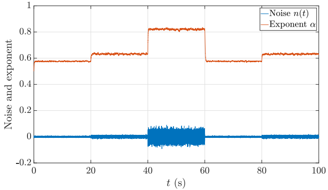
$$\sum_D : \begin{cases} z_1^+ = z_1 + hz_2^+ + h\lambda_1\mu|e_1|^\alpha \mathrm{sgn}(e_1) \\ z_2^+ = z_2 + h\lambda_2\mu^2\alpha|e_1|^{2\alpha-1}\mathrm{sgn}(e_1) \\ y_m = y + n(t) \end{cases}$$

- **HPF:** $Y_{mhf}(s) = \frac{s'^4}{(s'^2+0.7654s'+1)(s'^2+1.8478s'+1)} y_m,\ s' = \frac{s}{\omega_c}$
- **LPF:** $\dot{b}(t) = \tau(|y_{mhf}| - b(t))$
- **Variation:** $\alpha = 0.5\left(1 + \frac{b(t)}{b(t)+\varepsilon}\right)$
- **Adaption:** $\begin{cases} \alpha \to 0.5, & \text{if } b(t) \to 0, \\ \alpha \to 1, & \text{if } b(t) \to \infty. \end{cases}$

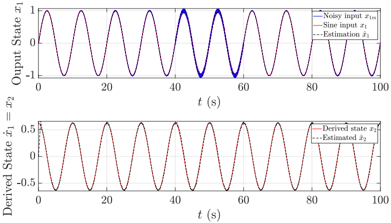
When using implicit-projector, $\tilde{u} : \begin{cases} \frac{\mu\lceil e_{1m}\rfloor^{(1-\alpha)}}{h\lambda}, & \mu|e_{1m}|^{(1-\alpha)} < h\lambda \\ \mathrm{sgn}(e_{1m}), & |e_{1m}| < h\lambda \end{cases}$

Levant's variable exponent (*sgn* function)

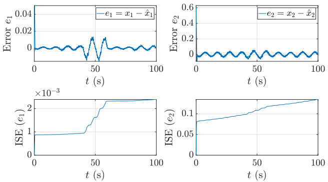
$$\Sigma_D : \begin{cases} \dot{z}_1 = z_2 + \lambda_1 |e_1|^{\alpha} \text{sgn}(e_1) \\ \dot{z}_2 = \lambda_2 \ \text{sgn}(e_1) \\ \alpha = 0.5\left(1 + \frac{b}{b+\varepsilon}\right) \\ x_{1m} = x_1 + n \\ e_1 = x_{1m} - z_1 \end{cases}$$



Exponent



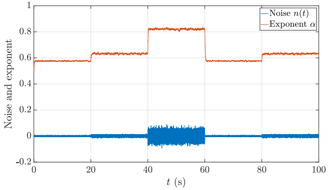
Estimations



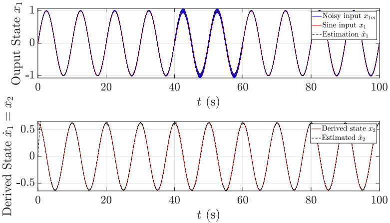
Errors

Levant's variable exponent (*proj* function)

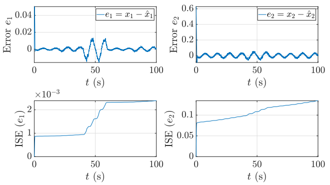
$$
\Sigma_D : \begin{cases} \dot{z}_1 = z_2 + \lambda_1 |e_1|^{\alpha} \tilde{u} \\ \dot{z}_2 = \lambda_2 \tilde{u} \\ \alpha = 0.5\left(1 + \frac{b}{b+\varepsilon}\right) \\ x_{1m} = x_1 + n \\ e_1 = x_{1m} - z_1 \end{cases}
$$



Exponent



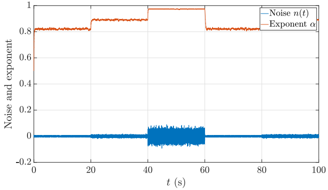
Estimations



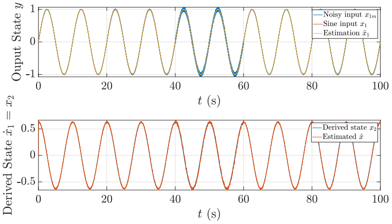
Errors

Semi-Implicit variable exponent

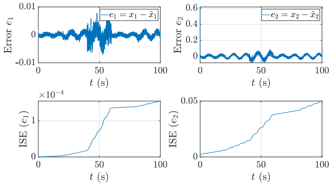
$$\sum_D : \begin{cases} z_1^+ = z_1 + hz_2 + h\lambda_1\mu|e_1|^{\alpha}\tilde{u} \\ z_2^+ = z_2 + h\lambda_2\mu^2\alpha|e_1|^{2\alpha-1}\tilde{u} \\ \alpha = 0.5\left(1 + \frac{b}{b+\varepsilon}\right) \\ x_{1m} = x_1 + n \\ e_1 = x_{1m} - z_1 \end{cases}$$



Exponent



Estimations



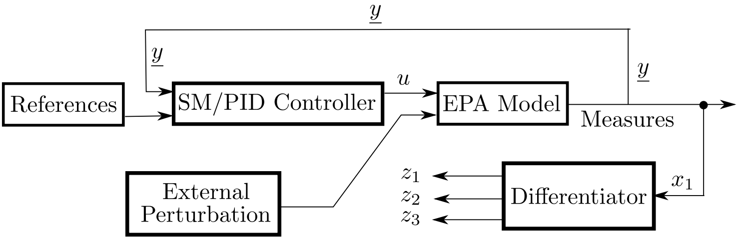
Errors

Summary

Mean, Max and Integral Square Errors (rounded off to one decimal place) for the differentiators of sinusoidal signal

Differentiator	λ_1	λ_2	μ	f (Hz)	**Mean**		**Max**		**SSE**	
					e_1	e_2	e_1	e_2	e_1	e_2
Levant vary	2	1	N/A	0.1	$1.7\ 10^{-4}$	$1.7\ 10^{-3}$	0.05	0.05	$2.4\ 10^{-3}$	0.13
	6	3			$-1.7\ 10^{-5}$	$5.3\ 10^{-4}$	0.01	0.01	$2.1\ 10^{-4}$	0.1
Levant (Script, Simulink)	6	6	N/A	0.1	$-0.1\ 10^{-3}$	$0.4\ 10^{-3}$	0.01	0.6	$2\ 10^{-3}$	0.03
					10^{-3}	$0.4\ 10^{-3}$	0.01	0.6	$2\ 10^{-3}$	1.5
Semi-Implicit (script)	2	1	8	0.5	$1.1\ 10^{-4}$	0.02	0.2	3.1	$2.0\ 10^{-5}$	0.3
	12	6			$2.9\ 10^{-6}$	$2.5\ 10^{-3}$	0.2	3.2	$2.0\ 10^{-6}$	0.1
Semi-Implicit (script)	12	6	8	0.1	$2.2\ 10^{-6}$	$8.5\ 10^{-5}$	0.2	0.6	$2.0\ 10^{-6}$	0.01
Semi-Implicit (simulink)	1	0.5	16	0.1	$-8.1\ 10^{-6}$	$7.8\ 10^{-5}$	0.01	0.6	$1.5\ 10^{-4}$	0.05

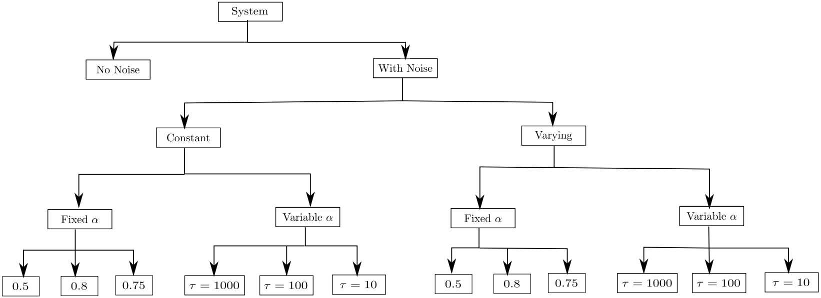
Scheme



Scheme of the simulator with a differentiator

First Analysis - Hierarchy

(To compare fixed/variable exponent semi-implicit differentiators by varying α and τ resp.)



Hierarchy of the initial analysis of differentiators

First Analysis - Differentiator

- Semi-Implicit Differentiator:

$$\begin{cases} z_1^+ = z_1 + hz_2^+ - h\lambda_1\mu|e_{1m}|^\alpha\mathcal{N}(e_{1m}) \\ z_2^+ = z_2 - h\lambda_2\mu^2\alpha|e_{1m}|^{2\alpha-1}\mathcal{N}(e_{1m}) \\ y_m = y + n(t) \end{cases}$$

- Exponent variation structure:

$$\begin{cases} \alpha \to 0.5, \ \text{if } b(t) \to 0, \\ \alpha \to 1, \ \text{if } b(t) \to \infty. \end{cases}$$

First Analysis - Summary

Comparison of constant and varying noise with fixed α

Noise $n(t)$	α	**Implicit Differentiator** $y_m,\ z_1$	**Implicit Differentiator** $v_m,\ z_2$	**Max**$(\lvert n(t)\rvert)$ **from** z_1	**Max**$(\lvert n(t)\rvert)$ **from** z_2
Constant for 10 s as $0.005n(t)$	0.5	Close estimation and with less distortions.	Low distortion but offset prevails.	$1.3\ 10^{-3}$	$0.9\ 10^{-3}$
	0.8	Smoother curve with visible offset	Smooth curve with high offset observed.	$4.2\ 10^{-4}$	10^{-4}
	0.75	Quicker convergence with state and low offset.	No visible changes observed.	$5\ 10^{-4}$	$1.3\ 10^{-4}$
Varying for 50 s as $[0.005n(t)$, $0.01n(t)$, $0.05n(t)$, $0.005n(t)$, $0.01n(t)]$	0.5	Distorted curve, especially at higher noise.	Distorted but smaller offset exists.	10^{-3}, $1.5\ 10^{-3}$ $3.5\ 10^{-3}$, 10^{-3}, $1.5\ 10^{-3}$	10^{-3}
	0.8	Smooth curve with better estimation.	Smoother curve with bigger offset observed.	$1.2\ 10^{-3}$, $1.7\ 10^{-3}$ $3.7\ 10^{-3}$, $1.2\ 10^{-3}$, $2\ 10^{-3}$	10^{-3}
	0.75	Slightly quicker convergence with state.	No visible changes observed.	$0.4\ 10^{-3}$, $0.6\ 10^{-3}$ $2.5\ 10^{-3}$, $0.5\ 10^{-3}$, $0.9\ 10^{-3}$	10^{-4}, $1.5\ 10^{-4}$ $3.5\ 10^{-3}$, $1.2\ 10^{-4}$, $2\ 10^{-4}$

First Analysis - Summary

Comparison of constant and varying noise with variable α

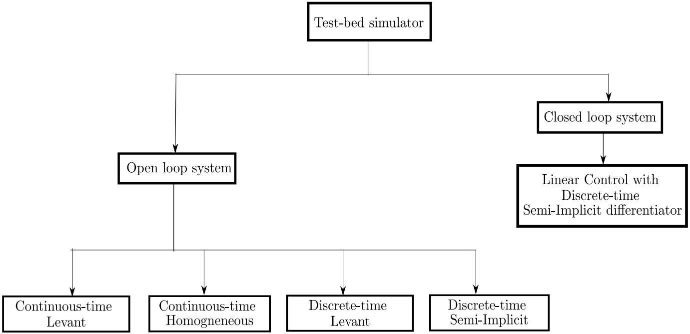
Noise $n(t)$	τ	**Implicit Differentiator** y_m, z_1	**Implicit Differentiator** v_m, z_2	α **range**	**Max**$(\lvert n(t)\rvert)$ **from** z_1 (10^{-3})	**Max**$(\lvert n(t)\rvert)$ **from** z_2 (10^{-4})
Const	1000	Better estimation with no visible offset observed	Smoother curve and better estimation.	0.6 – 0.7	1.3	0.1
	100			0.6	1	6
	10			≈ 0.6	0.9	4.8
Vary	1000	Clearly better estimation observed with less offset.	Good estimation with offset only in higher noise range.	0.6, 0.7 0.9, 0.6,0.7	1, 1.2 2.5, 1, 1.2	6, 4 3, 6, 4
	100			0.7, 0.8, 0.9 0.7, 0.8	0.8, 1 3.7, 0.8, 1	5, 4 3, 5, 4
	10			0.6, 0.7, 0.8, 0.6, 0.7	0.6, 0.8 2.3, 0.6, 0.8	5, 4 3, 5, 4

Interpretations:

- **Hits:**
  - Fixed exponent differentiator – $\alpha = 0.75$ shows good estimation.
  - Variable exponent differentiator – encouraging.
  - Faster adaption of α with increasing τ.
  - Increasing τ also increases oscillation range of α.
- **Misses:**
  - Uses the measure of position controlled by the second-order implicit control – Needs more attention...!
  - A clear offset visible in the velocity estimate z_2 – probably due to chattering in the controlled velocity.
  - Not able to proceed with estimation of acceleration as velocity estimate is not accurate.

Second Analysis - Hierarchy

(To compare the differentiators without controller and with a linear controller.)



Hierarchy of the re-analysis of differentiators

Second Analysis - Cont. time variable exponent Differentiators

Open-loop:

- Levant: $\begin{cases} \dot{z}_1 = z_2 + \lambda_1 |e_1|^{\alpha} \mathrm{sgn}(e_1) \\ \dot{z}_2 = \lambda_2 \mathrm{sgn}(e_1) \\ e_1 = y_m - z_1 \end{cases}$
- Homogeneous: $\begin{cases} \dot{z}_1 = z_2 + \lambda_1 \mu |e_1|^{\alpha} \mathrm{sgn}(e_1) \\ \dot{z}_2 = \lambda_2 \mu^2 \alpha |e_1|^{2\alpha - 1} \mathrm{sgn}(e_1) \\ e_1 = y_m - z_1 \end{cases}$

Second Analysis - Disc. time variable exponent Differentiators

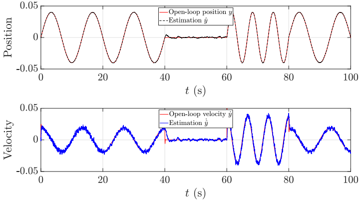
Open loop:

- Levant: $\begin{cases} z_1^+ = z_1 + hz_2 + h\lambda_1 |e_1|^\alpha \text{sgn}(e_1) \\ z_2^+ = z_2 + h\lambda_2 \text{sgn}(e_1) \\ e_1 = y_m - z_1 \end{cases}$
- Semi-Implicit: $\begin{cases} z_1^+ = z_1 + hz_2 + h\lambda_1 \mu |e_1|^\alpha \tilde{u} \\ z_2^+ = z_2 + h\lambda_2 \mu^2 \alpha |e_1|^{2\alpha-1} \tilde{u} \\ e_1 = y_m - z_1 \end{cases}$

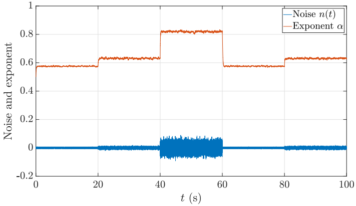
Closed loop:

- Control: Linear PID with control input $u_{eq} = -k_1 e_1 - k_2 \dot{e}_1 - k_3 \ddot{e}_1$
- Estimation: Discrete-time Semi-implicit variable exponent

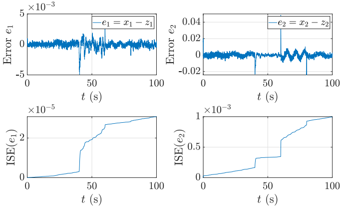
Second Analysis - Results from closed loop test



(a) States and estimations



(b) Noise and varying exponent



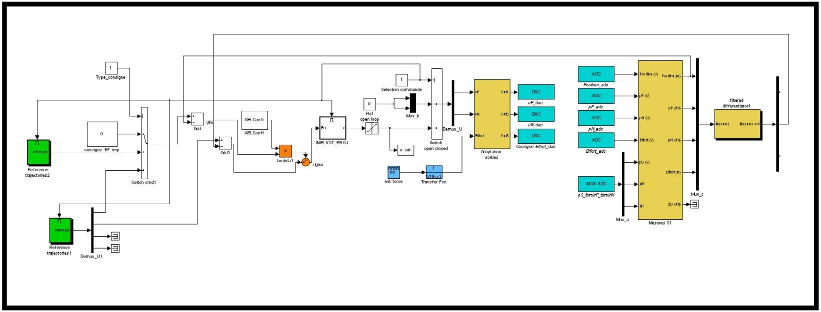
(c) Estimation Errors and ISEs

Second Analysis - Summary

Mean and SSE of estimation errors in the simulator

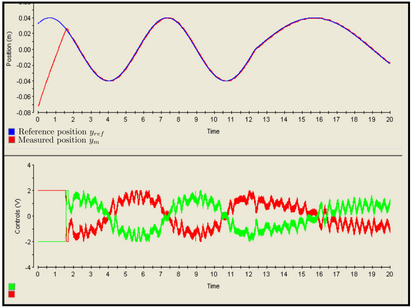
Loop	Differentiator	λ_1	λ_2	μ	Mean		SSE	
					e_1	e_2	e_1	e_2
Open	α varying (Cont.) (6.27)	1	0.4	1	$-4.1\ 10^{-6}$	$7.7\ 10^{-8}$	$9.5\ 10^{-4}$	$4.6\ 10^{-3}$
	Levant (Cont.) (6.29)	0.4	0.1	N/A	$-2.1\ 10^{-5}$	$-1.1\ 10^{-5}$	$1.5\ 10^{-3}$	$8.5\ 10^{-3}$
	Levant (Disc.) (6.31)	0.2	0.06	N/A	$-1.9\ 10^{-5}$	$-1.2\ 10^{-5}$	$1.6\ 10^{-5}$	$8.3\ 10^{-5}$
	Semi-Implicit (6.33)	1	0.6	0.7	$-1.8\ 10^{-6}$	$6.3\ 10^{-6}$	$8.2\ 10^{-4}$	$4\ 10^{-3}$
Closed	Semi-Implicit (6.2.5)	1	0.8	1.2	$-4.8\ 10^{-5}$	$-1.1\ 10^{-4}$	$3.1\ 10^{-3}$	0.1

Implementation on the EPA



Second order Implicit controller designed for the real-time interface

Results



Position (top) and control input (bottom)

Remarks:

- Good control on the position as seen in the simulator.
- A clear control input instead of *bang-bang-like* in explicit.

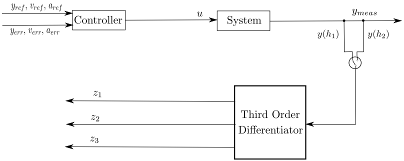
Differentiators

Only fixed exponent homogeneous differentiators are considered for analysis [2]:

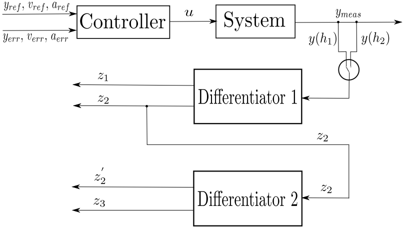
- Third-order Levant's differentiator
- Two cascaded second-order differentiators
  - Explicit Euler Discretization (E2D) method
  - Semi-Implicit Discretization based on Explicit *sgn* function (SIDES) method
  - Semi- Implicit Discretization based on pseudo Linearization (SIDL) method
  - Semi-Implicit Discretization based on implicit *sgn* Projector (SIDP) method
  - Semi-Implicit Discretization based on implicit *sgn* Modified Projector (SIDMP) method

[2] Michel et al, *"An experimental investigation of discretized homogeneous differentiators: pneumatic actuator case"*, IEEE Journal of Emerging and Selected Topics in Industrial Electronics, 2020 (submitted)

Schematic representation



(a) Third-order Levant differentiator



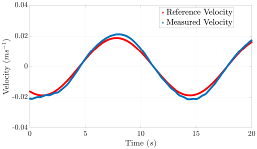
(b) Two cascaded differentiators

Gain Settings

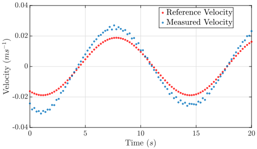
Differentiators	λ_1	λ_2	λ_3	λ_4	α_1	α_2
Third-order Levant	1.5	0.625	0.625	N/A	0.7	N/A
E2D, SID-L/P/MP	1.5	0.625	1.5	0.625	0.75	0.7
SIDES	0.25	0.025	0.25	0.025	0.5	0.5

Third-order Levant

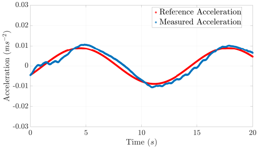
$$\Sigma_D : \begin{cases} z_1^+ = z_1 + h(\lambda_1 \mu \lceil e_1 \rfloor^{\alpha} + z_2) \\ z_2^+ = z_2 + h(\lambda_2 \mu^2 \lceil e_1 \rfloor^{2\alpha-1} + z_3) \\ z_3^+ = z_3 + h(\lambda_3 \mu^3 \lceil e_1 \rfloor^{3\alpha-2}) \end{cases}$$



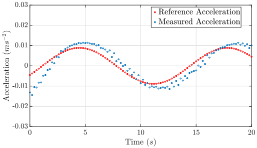
Velocity $h = 0.2$ ms



Velocity $h = 0.2$ s



Acceleration $h = 0.2$ ms



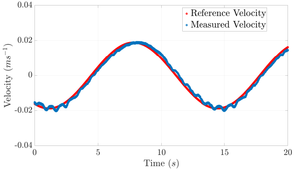
Acceleration $h = 0.2$ s

E2D method

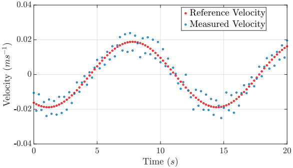
$$\Sigma_{D1} : \begin{cases} z_1^+ = z_1 + h(\lambda_1 \lceil e_1 \rfloor^{\alpha_1} + z_2) \\ z_2^+ = z_2 + h(\lambda_2 \lceil e_1 \rfloor^{2\alpha_1 - 1}) \end{cases}$$

$$\Sigma_{D2} : \begin{cases} {z_2'}^+ = z_2' + h(\lambda_3 \lceil e_2 \rfloor^{\alpha_2} + z_3) \\ z_3^+ = z_3 + h(\lambda_4 \lceil e_2 \rfloor^{2\alpha_2 - 1}) \end{cases}$$

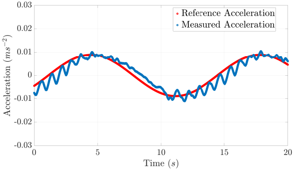
Estimations



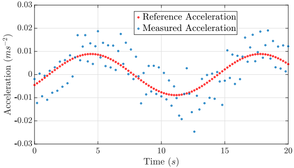
(a) $h = 0.2\ ms$



(b) $h = 0.2\ s$



(c) $h = 0.2\ ms$



(d) $h = 0.2\ s$

Velocity (top) and acceleration (bottom)

SIDES method [Polyakov et al., 2014]

$$\Sigma_D : \begin{cases} z_1^+ = z_1 + h(\lambda_1 |e_1^+|^{\alpha_i} \mathrm{sgn}(y_1 - z_1) + z_2^+) \\ z_2^+ = z_2 + h(\lambda_2 |e_1^+|^{2\alpha_i - 1} \mathrm{sgn}(y_1 - z_1)) \end{cases}$$

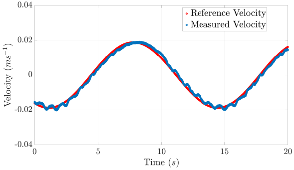
Solving for $\alpha = 0.5$:

$$\Sigma_{D1} : \begin{cases} z_1^+ = z_1 + h\lambda_1 \, w_1 \, \mathrm{sgn}(e_1) + hz_2 \\ z_2^+ = z_2 + h\lambda_2 \mathrm{sgn}(e_1) \end{cases}$$

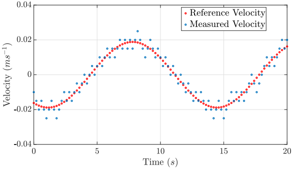
$$\Sigma_{D2} : \begin{cases} z_2'^+ = z_2' + h\lambda_3 \, w_2 \, \mathrm{sgn}(e_2) + hz_3 \\ z_3^+ = z_3 + h\lambda_4 \mathrm{sgn}(e_2) \end{cases}$$

with $w_2 = \frac{-h\lambda_3 + \sqrt{(h\lambda_3)^2 + 4|e_2|}}{2}$

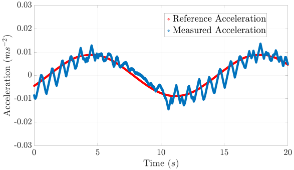
Estimations with SIDES



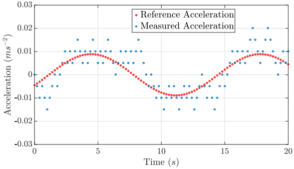
(a) $h = 0.2\ ms$



(b) $h = 0.2\ s$



(c) $h = 0.2\ ms$



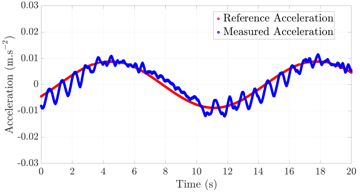
(d) $h = 0.2\ s$

Velocity (top) and acceleration (bottom)

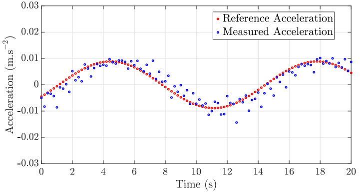
SIDL [Wetzlinger et al., 2019]

Replacing $\text{sgn}(e_1)$ by $\frac{e_1}{|e_1|}$:

$$\Sigma_D : \begin{cases} z_1^+ = z_1 + h(\lambda_1 |e_1|^{\alpha_1 - 1} e_1^+ + z_2^+) \\ z_2^+ = z_2 + h(\lambda_2 |e_1|^{2(\alpha_1 - 1)} e_1^+) \end{cases}$$



(a) $h = 0.2\ ms$



(b) $h = 0.2\ s$

Acceleration estimate

SIDP

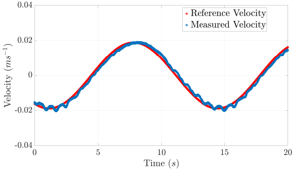
$$\Sigma_{D1} : \begin{cases} z_1^+ = z_1 + h\left(z_2^+ + \lambda_1 |e_1|^{\alpha_1} \mathcal{N}(e_1, \alpha_1, \lambda_1)\right) \\ z_2^+ = z_2 + h\left(\lambda_2 |e_1|^{2\,\alpha_1 - 1} \mathcal{N}(e_1, \alpha_1, \lambda_1)\right) \end{cases}$$

$$\Sigma_{D2} : \begin{cases} z_2^{\prime +} = z_2 + h\left(z_3^+ + \lambda_3 |e_2|^{\alpha_2} \mathcal{N}(e_2, \alpha_2, \lambda_3)\right) \\ z_3^+ = z_3 + h\left(\lambda_4 |e_2|^{2\,\alpha_2 - 1} \mathcal{N}(e_2, \alpha_2, \lambda_3)\right) \end{cases}$$

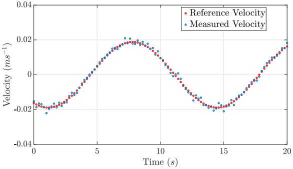
The projector is given by:

$$\mathcal{N}(e_i, \alpha_i, \lambda_j) := \begin{cases} \dfrac{\lceil e_i \rfloor^{1-\alpha_i}}{\lambda_j h} \;, & |e_i|^{1-\alpha_i} < \lambda_j h \\ \operatorname{sgn}(e_i) \;, & |e_i|^{1-\alpha_i} \geq \lambda_j h \end{cases}$$

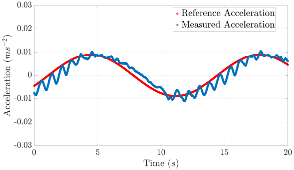
Estimations with SIDP



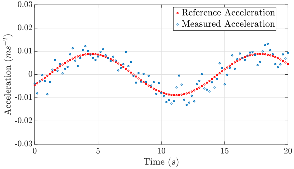
(a) $h = 0.2\ ms$



(b) $h = 0.2\ s$



(c) $h = 0.2\ ms$



(d) $h = 0.2\ s$

Velocity (top) and acceleration (bottom)

SIDMP

$$\mathcal{N}_\theta(e_i, \alpha_i, \lambda_j) := \begin{cases} \frac{(1-\theta)\lceil e_i \rfloor^{1-\alpha_i}}{\lambda_j h} & , \quad (1-\theta)|e_i|^{1-\alpha_i} < \lambda_j h \\ \text{sgn}(e_i) & , \quad |e_i|^{1-\alpha_i} \geq \lambda_j h \end{cases}$$

Before Modification:

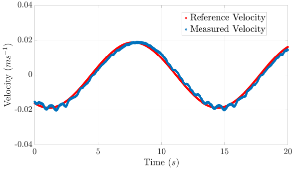
$\mathcal{N} \to 0 = e_i - h\lambda_j |e_i|^{\alpha_i} \mathcal{N}(e_i, \alpha_i, \lambda_j)$

After modification:

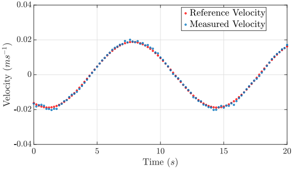
$\mathcal{N}_\theta \to \theta e_i = e_i - h\lambda_j |e_i|^{\alpha_i} \mathcal{N}_\theta(e_i, \alpha_i, \lambda_j)$

$$\boxed{\theta \text{ is set to } \frac{1}{2}}$$

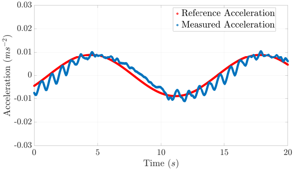
Estimations with SIDMP



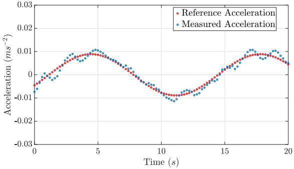
(a) $h = 0.2\ ms$



(b) $h = 0.2\ s$



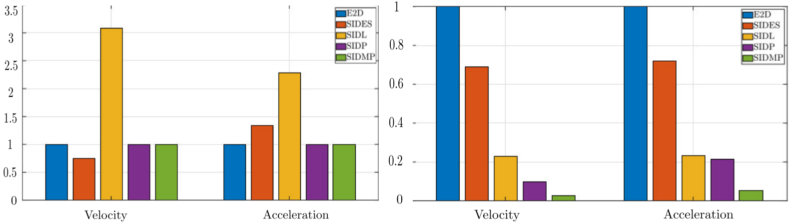
(c) $h = 0.2\ ms$



(d) $h = 0.2\ s$

Velocity (top) and acceleration (bottom)

Summary



(a) $h = 0.2\ ms$ (b) $h = 0.2\ s$

Pictorial representation of Normalized SSE

Remarks:

- Performances differ for higher sampling period.
- SIDMP shows the best performance among the compared methods.

Control

- Second-order control law designed and tested on the simulator.
- Experimental results show encouraging results but needs more tuning.
- Explicit control input seems more like a *bang-bang* but reduced in the case in implicit method.
- Design of a third-order control law – in perspective.

Estimation

- Explicit, implicit and semi-implicit differentiators were designed.
- Semi-implicit method resulted in efficient estimations.
- SIDMP with a modified projector exhibited even better results.
- Variable exponent differentiators were tested on the simulator, yet to be implemented on the test-bed.
- Estimations were carried out using the position measure controlled by the explicit SMC method – aiming to use the measure controlled by the implicit method.

Future objectives

- To obtain better tuned results with the second-order implicit SMC.
- To design and test the third-order implicit control law.
- To implement the adaptive differentiators on the system.
- To close the system's loop by making it achieve an *observer-based control.*

Thank you for the attention!

Comments & Questions?

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pages 5995–6000.

List of Abbreviations

List 1:

- EPA – Electro-pneumatic Actuator
- SM – Sliding Mode
- DSM - Discrete-time Sliding Mode
- SMC - Sliding Mode Control
- DSMC - Discrete-time Sliding Mode Control
- SMD - Sliding Mode Differentiators
- HOSMC - Higher Order SMC
- HDSM - Higher Order Discrete-time Sliding Mode
- HOMD - Homogeneous Differentiator

↑

List of Abbreviations

- E2D - Explicit Euler Discretization
- SIDES - Semi-Implicit Discretization based on Explicit *sgn* function
- SIDL - Semi-Implicit Discretization based on psuedo-Linearization
- SIDP - Semi-Implicit Discretization based on implicit *sgn* function with Projector
- SIDMP - Semi-Implicit Discretization based on implicit *sgn* function with Modified Projector

↑

Nomenclature

- x_i – System states
- z_i – Estimated states
- h – Sampling period
- $(\bullet)_{ref}$ – References
- u – Control input to the system
- $\tilde{u}$ – Correction term of the differentiator
- $\mathcal{N}_{[-1,1]}(\bullet)$ – Projector output/inverse of sgn function
- k_i – Controller gains
- λ_i – Differentiator gains
- μ – Parameter in differentiator to cancel perturbation

↑

Nomenclature

- e_i – Error in traction/estimation
- σ – Sliding variable
- $(\bullet)^+$ – values at the instant of $(k+1)h$
- $(\bullet)^-$ – values at the instant of $(k-1)h$
- $\lceil\bullet\rfloor^\gamma = |\bullet|^\gamma sgn(\bullet)$
- ISE $(\bullet)$ – Integral Square Error of $(\bullet)$
- Max $(\bullet)$ – Maximum of $(\bullet)$
- Mean $(\bullet)$ – Average of $(\bullet)$
- SSE $(\bullet)$ – Sum of Square Error of $(\bullet)$

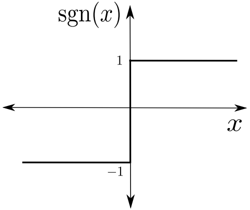
$\uparrow\uparrow$

Nomenclature

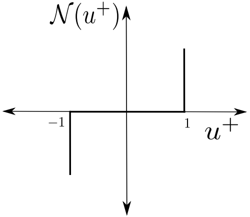
- $Y_{mhf}(s)$ – Fourth-order Butterworth High-pass filter Transfer function
- $b(t)$ – First-order Low-pass filter
- $s' = \frac{s}{\omega_c}$ where ω_c is the cut-off frequency
- τ – time-constant of the first-order LPF
- $\lceil \bullet \rfloor^{\gamma} = | \bullet |^{\gamma} sgn(\bullet)$
- α – Homogeneous exponent term
- ε – a very small positive parameter helping in α variation

↑

Application



(a) sgn(•) function



(b) Normal Cone $\mathcal{N}(\bullet)$

Recall:

$$\begin{cases} x^+ = x - Khu^+, \\ u^+ = \text{sgn}(x^+) \end{cases}$$

$$\implies u^+ = \text{sgn}(x - Khu^+)$$

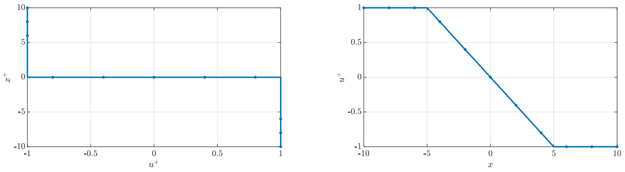
$$\implies \mathcal{N}_{[-1,1]}(u^+) = x - Khu^+$$

$$\implies \boxed{x - Khu^+ - \mathcal{N}_{[-1,1]}(u^+) = 0}$$

$\uparrow$

Solution

$$u^+ = \begin{cases} \frac{x}{h}, & |x| < h, \\ \mathrm{sgn}(x), & \text{elsewhere.} \end{cases}$$



(a) $-\mathcal{N}_{[-1,1]}(u^+)$ (b) u^+ from projector

Illustration of the implicit methodology

Solving...!

$\uparrow$ ***Case (i):*** $\mathcal{N}_{[-1,1]}(u^+) = \mathbb{R}^-$

$$x - Kh(-1) - \mathbb{R}^- \to 0 \implies \frac{x}{Kh} + 1 < 0 \implies \boxed{\frac{x}{Kh} < -1}$$

Case(ii): $\mathcal{N}_{[-1,1]}(u^+) = \mathbb{R}^+$

$$x - Kh(1) - \mathbb{R}^+ \to 0 \implies \frac{x}{Kh} - 1 > 0 \implies \boxed{\frac{x}{Kh} > 1}$$

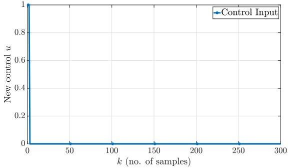
Case(iii): $\mathcal{N}_{[-1,1]}(u^+) = 0$

$$x - Khu^+ - 0 \to 0 \implies \frac{x}{Kh} - u^+ = 0 \implies \boxed{u^+ = \frac{x}{Kh}}$$

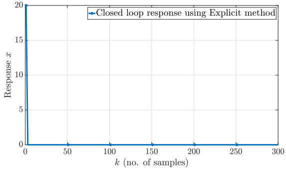
Variables

- y – position of the pneumatic actuator (or piston)
- v – actuator linear velocity
- $\dot{v} = a$ – actuator acceleration
- u – control input (or simply input)
- F_{ext} – external perturbation from the perturbation actuator
- p_P, p_N – pressures in chambers P and N
- r – ideal gas constant
- b_v – viscosity coefficient
- S – Useful surface area of the cylinder
- T – Temperature (in K)
- k – Polytropic coefficient
- M – Nominal mass of all the mobile parts

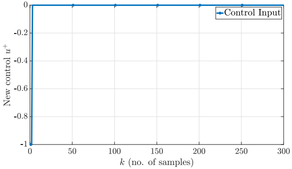
$x_0 = 20$



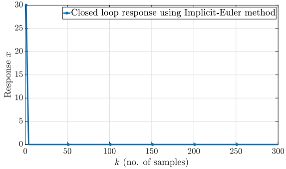
(a) Explicit Control u



(b) Explicit Output x



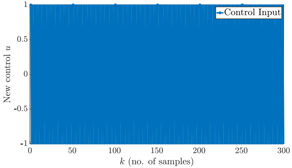
(c) Implicit Control u^+



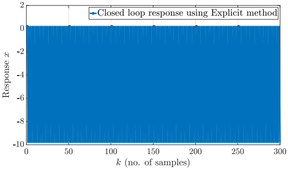
(d) Implicit Output x

Explicit (top) and Implicit (bottom) Controls with $x_0 = 20$

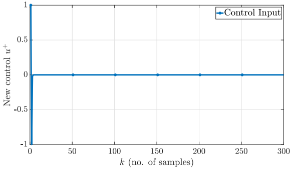
$x_0 = 0.2$



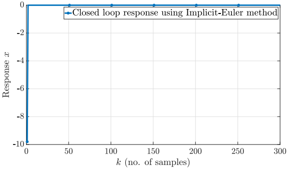
(a) Explicit Control u



(b) Explicit Output x



(c) Implicit Control u^+

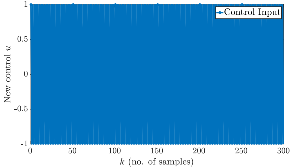


(d) Implicit Output x

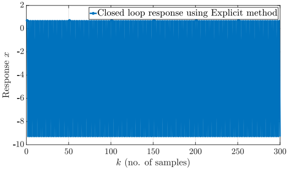
Explicit (top) and Implicit (bottom) Controls with $x_0 = 0.2$

$x_0 = 0.7$

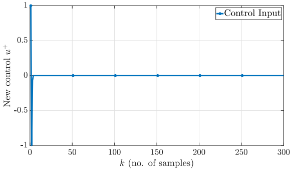
↑



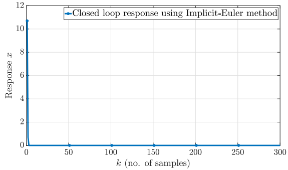
(a) Explicit Control u



(b) Explicit Output x



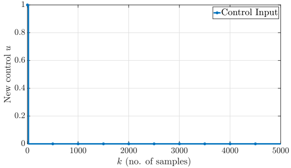
(c) Implicit Control u^+



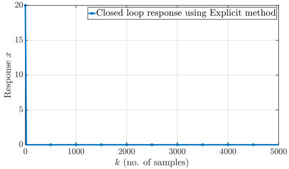
(d) Implicit Output x

Explicit (top) and Implicit (bottom) Controls with $x_0 = 0.7$

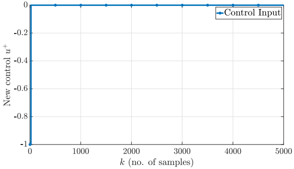
$h = 1$



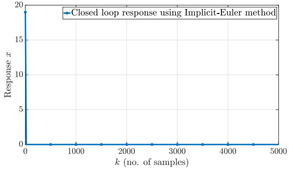
(a) Explicit Control u



(b) Explicit Output x



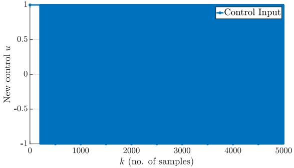
(c) Implicit Control u^+



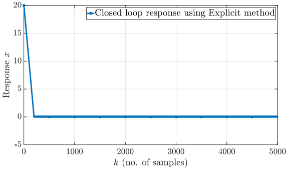
(d) Implicit Output x

Explicit (top) and Implicit (bottom) Controls with $h = 1$

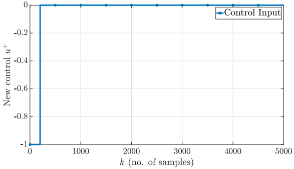
$h = 0.1$



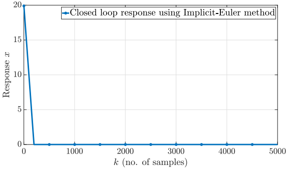
(a) Explicit Control u



(b) Explicit Output x



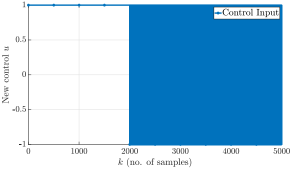
(c) Implicit Control u^+



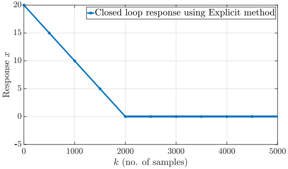
(d) Implicit Output x

Explicit (top) and Implicit (bottom) Controls with $h = 0.1$

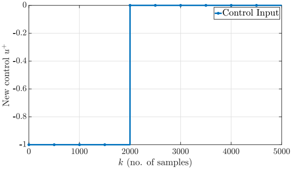
$h = 0.01$



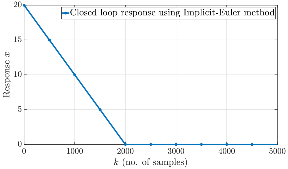
(a) Explicit Control u



(b) Explicit Output x



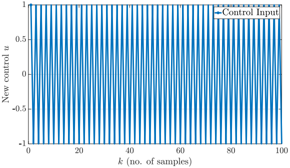
(c) Implicit Control u^+



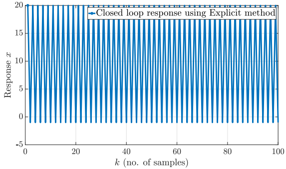
(d) Implicit Output x

Explicit (top) and Implicit (bottom) Controls with $h = 0.01$

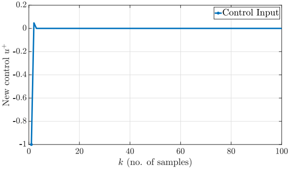
$h = 21$



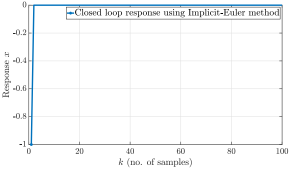
(a) Explicit Control u



(b) Explicit Output x



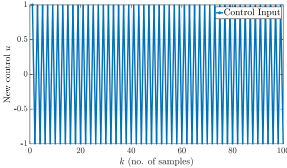
(c) Implicit Control u^+



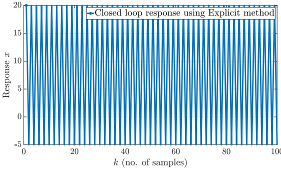
(d) Implicit Output x

Explicit (top) and Implicit (bottom) Controls with $h = 25$

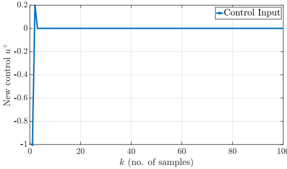
$h = 25$



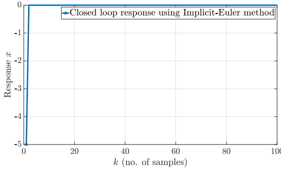
(a) Explicit Control u



(b) Explicit Output x



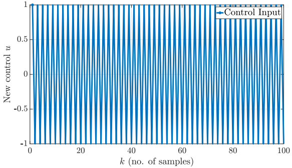
(c) Implicit Control u^+



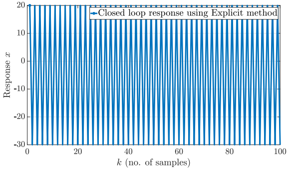
(d) Implicit Output x

Explicit (top) and Implicit (bottom) Controls with $h = 25$

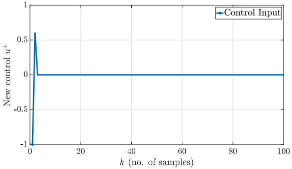
$h = 50$



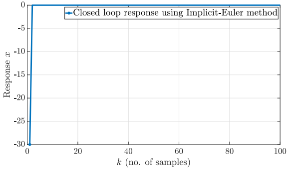
(a) Explicit Control u



(b) Explicit Output x



(c) Implicit Control u^+

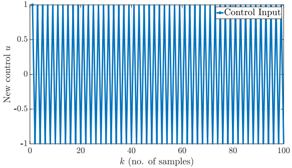


(d) Implicit Output x

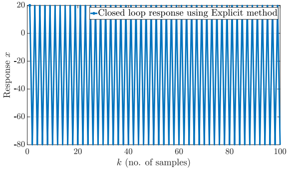
Explicit (top) and Implicit (bottom) Controls with $h = 50$

$h = 100$

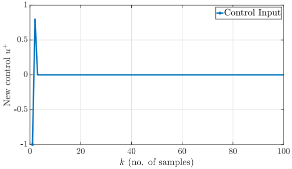
↑



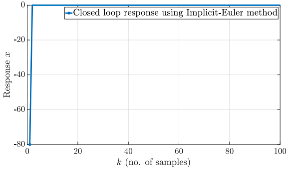
(a) Explicit Control u



(b) Explicit Output x



(c) Implicit Control u^+



(d) Implicit Output x

Explicit (top) and Implicit (bottom) Controls with $h = 100$

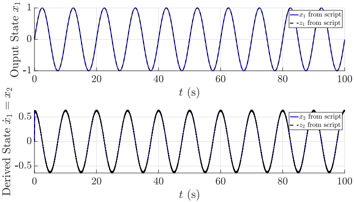
Influence of Gains

Table: Influence of gains on the estimation errors and their ISE indices

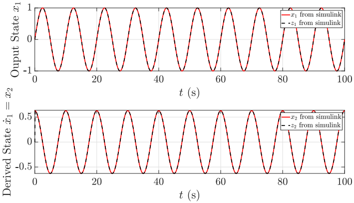
	λ_1	λ_2	e_{1max}	e_{2max}	ISE(e_1)	ISE (e_2)
Script	6	6	0.0091	0.6283	0.0020	0.0328
Simulink	6	6	0.0057	0.6283	0.0020	1.4991
Script	6	10 ↑	0.0090 ↓	0.6283	0.0020	0.0274 ↓
Simulink			0.0071 ↑	0.6283	0.0002 ↓	1.2082 ↓
Script	6	3 ↓	0.0091	0.6283	0.0020	0.0462 ↑
Simulink			0.0008 ↓	0.6283	0.0004 ↓	2.8103 ↑
Script	10 ↑	6	0.0068 ↓	0.6283	0.0020	0.0332 ↑
Simulink			0.0033 ↓	0.6283	0.0001 ↓	1.7238 ↑
Script	12 ↑	6	0.0066 ↓	0.6283	0.0020	0.0336 ↑
Simulink			0.0028 ↓	0.6283	0.0000 ↓	1.8738 ↑

Green – the setting for which the simulation is performed

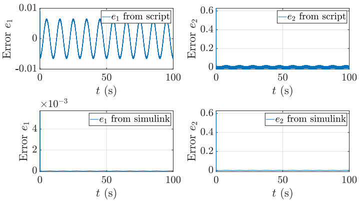
Results ↑



(a) Script



(b) Simulink



(c) Estimation Errors

Semi-Implicit Homogeneous Euler Differentiator for a Second-Order System

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Annual DigitSlid meeting - Thursday 10th September 2020

Outline

- Introduction
- Problem statement
- Semi-implicit homogeneous Euler differentiator
- Numerical results
- Conclusion

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Introduction

- One way to treat real-time differentiation is to use numerical based Euler approximations
- Recently, based on the Acary & Brogliato's *implicit* framework, a semi-implicit discretization has been proposed to deal with homogeneous control
- We propose in this work a semi-implicit based homogeneous differentiator to take benefits from the Euler approximation and the implicit framework

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Problem statement

Continuous time system Let be $p(t)$ a bounded perturbation, which is considered unknown. The system under study consists of a double integrator of the form

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = p(t) \\ y = x_1 \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^2$ is the state of the system, $y \in \mathbb{R}$ is the output of the system

Homogeneous continuous time differentiator

$$\begin{cases} \dot{z}_1 = z_2 + \lambda_1 \mu \lceil \epsilon_1 \rfloor^{\alpha} \\ \dot{z}_2 = \lambda_2 \mu^2 \lceil \epsilon_1 \rfloor^{2\alpha - 1} \mathrm{sgn}(\epsilon_1) \\ \hat{y} = z_1 \end{cases} \tag{2}$$

where $\epsilon_1 = x_1 - z_1$ including the notation $\lceil \bullet \rfloor^{\alpha} = | \bullet |^{\alpha} \mathrm{sgn}(\bullet)$

- $\lambda_i > 0$, $i = 1, 2$ allow to have the eigenvalues of the differentiation error ϵ_1 sufficiently stables
- the coefficient μ is chosen sufficiently large to cancel the effect of the unknown perturbation $p(t)$

Problem statement

The corresponding *Implicit Euler discrete-time system* reads

$$\begin{cases} x_1^+ = x_1 + h\,x_2^+ = x_1 + h(x_2 + hp^+) \\ x_2^+ = x_2 + h\,(p^+) \end{cases} \tag{3}$$

where h is the sampling-time and assuming that

1. there exist $\dot{y}_M > 0$, such that for all $t > 0$, $|\dot{y}(t)| < \dot{y}_M$
2. the perturbation $p(t)$ is a constant parameter or slowly variable, this implies that for sufficient small $h > 0$, $p^+ \approx p$

Goal : The objective is to give an Euler discretization of the continuous-time homogeneous second-order differentiator

- First solution :

Explicit homogeneous Euler differentiator

$$\begin{cases} \hat{x}_1^+ = \hat{x}_1 + h\left(\hat{x}_2 + \lambda_1 \lceil e_1 \rfloor^{\alpha}\right) \\ \hat{x}_2^+ = \hat{x}_2 + h\left(\lambda_2 \lceil e_1 \rfloor^{2\alpha - 1}\right) \end{cases} \tag{4}$$

where $e_1 = x_1 - \hat{x}_1$

$\Longrightarrow$ This solution is not attractive since it suffers from chattering phenomena

Problem statement

- Second solution :

Implicit homogeneous Euler differentiator

$$\begin{cases} \hat{x}_1^+ = \hat{x}_1 + h\left(\hat{x}_2^+ + \lambda_1 \lceil e_1^+ \rfloor^{\alpha}\right) \\ \hat{x}_2^+ = \hat{x}_2 + h\left(\lambda_2 \lceil e_1^+ \rfloor^{2\alpha - 1}\right) \end{cases} \quad (5)$$

$\Longrightarrow$ When e_1^+ tends to zero, the estimated $\hat{x}_2$ is zero, therefore the implicit homogeneous Euler second-order differentiator does not work

Outline

Semi-implicit homogeneous Euler differentiator

The proposed *semi-implicit Euler discrete-time homogeneous differentiator* reads

$$\begin{cases} \hat{x}_1^+ = \hat{x}_1 + h\left(\hat{x}_2^+ + \lambda_1 |e_1|^{\alpha} \mathcal{N}\right) \\ \\ \hat{x}_2^+ = \hat{x}_2 + E_1^+ h\left(\lambda_2 |e_1|^{2\,\alpha-1} \mathcal{N}\right) \end{cases} \tag{6}$$

where $e_1 := y - \hat{x}_1 = x_1 - \hat{x}_1$ and

$$\mathcal{N} := \begin{cases} |e_1|^{1-\alpha} < \lambda_1 h \ \ (e_1^+ = 0) \rightarrow \mathcal{N} = \frac{\lceil e_1 \rfloor^{1-\alpha}}{\lambda_1 h} \\ \\ |e_1|^{1-\alpha} \geq \lambda_1 h \ \ (e_1^+ \neq 0) \rightarrow \mathcal{N} = \mathrm{sgn}(e_1) \end{cases} \tag{7}$$

(See the presentation of Subiksha for a description of the other existing solutions)

Semi-implicit homogeneous Euler differentiator

E_1 depends on the *stability domains* and is defined as follow

$$\begin{cases} \text{set } E_1 = 1 \quad \text{if} \quad e_1 \in SD \\ \\ \text{set } E_1 = 0 \quad \text{if} \quad e_1 \notin SD \end{cases} \tag{8}$$

where SD is defined by $SD = \{e_1 \,/\, |e_1| \leq (\lambda_1 h)^{\frac{1}{1-\alpha}}\}$

The differentiation error dynamic reads

$$\begin{cases} e_1^+ = e_1 + h\left(e_2^+ - \lambda_1 |e_1|^\alpha \mathcal{N}\right) \\ \\ e_2^+ = e_2 + h\left(p^+ - E_1^+ \lambda_2 |e_1|^{2\alpha-1} \mathcal{N}\right) \end{cases} \tag{9}$$

Semi-implicit homogeneous Euler differentiator

Theorem 1 : For $h > 0$ and $\alpha \in]0, 1[$, there exist $\lambda_1 > 0$ and $\lambda_2 > 0$ such that the differentiation error dynamics (9) converge asymptotically to

$$SD_{1,2} : \;=\; \{e_1, e_2 \,/\, e_1 \in SD_1 \text{ and } e_2 \in SD_2\} \text{ with}$$

$$SD_1 \;=\; \left\{ e_1 \,/\, |e_1| \leq 2h^{\frac{1}{\alpha}} \left(\frac{p_M \lambda_1}{\lambda_2}\right)^{\frac{1}{\alpha}} \right\} \qquad \boxed{e_1 = h\, e_2}$$

$$SD_2 \;=\; \left\{ e_2 \,/\, |e_2| \leq 2h^{\frac{1-\alpha}{\alpha}} \left(\frac{p_M \lambda_1}{\lambda_2}\right)^{\frac{1}{\alpha}} \right\}$$

assuming that the absolute value of x_2^+ is strictly greater than the maximum time derivative of y, i.e., $|x_2^+| > \dot{y}_M$

Sketch of the proof

The proof is done in two steps, firstly the convergence of e_1 and after that the convergence of e_2

1. if $e_1 \in SD$ and if $e_2 \neq 0$, e_1^+ verifies $e_1^+ = he_2^+$
2. then $e_2^+ = e_2 + h\left(p^+ - E_1^+ \lambda_2 |he_1|^{2\alpha-1}\left(\frac{\lceil e_1 \rfloor^{1-\alpha}}{\lambda_1 h}\right)\right)$ as $e_1 = he_2$ and $E_1 = 1$, then

$$e_2^+ = e_2 + h\left(p^+ - \frac{\lambda_2}{\lambda_1 h}\lceil he_2 \rfloor^{\alpha}\right)$$

3. we deduce SD_2 and since $e_1 = h\, e_2$, we deduce finally SD_1

Outline

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Let us consider the discretized system

$$\begin{cases} x_1^{\frac{+}{10}} = x_1 + h' \, x_2^{\frac{+}{10}} = x_1 + h'(x_2 + h' p^{\frac{+}{10}}) \\ \\ x_2^{\frac{+}{10}} = x_2 + h' \, (p^{\frac{+}{10}}) \end{cases} \tag{10}$$

where $h = 10h' = 0.025$ s and the perturbation is $p(t) = \sin(at)$, also $a = 1$ and $(x_1(0), x_2(0)) = (0.45, 0)$

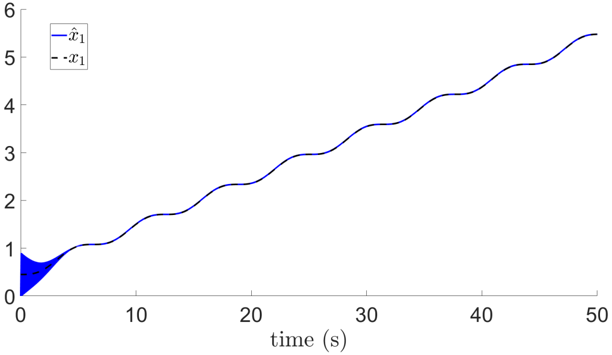
Semi-implicit homogeneous Euler differentiator

The semi-implicit and explicit homogeneous Euler differentiators are set with the set of parameters $\lambda_1 = 30$, $\lambda_2 = 5, \alpha = 0.6$

Remark : The parameters λ_1, λ_2 and α have been set in order to provide a fast dynamic to the observation as well as good tracking properties of the states x_1 and x_2

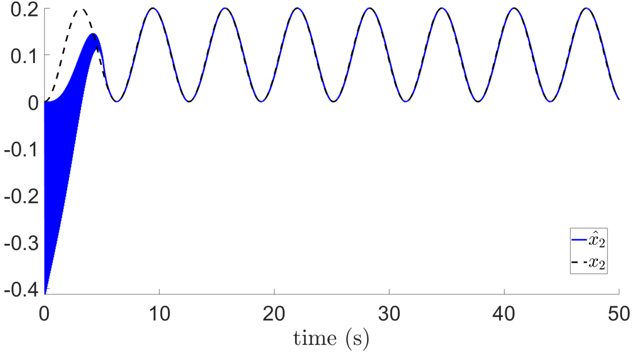
Semi-implicit homogeneous Euler differentiator

Numerical results - Semi-implicit differentiator



Semi-implicit differentiator State variable x_1 and estimated state variable $\hat{x}_1$ versus time (s)

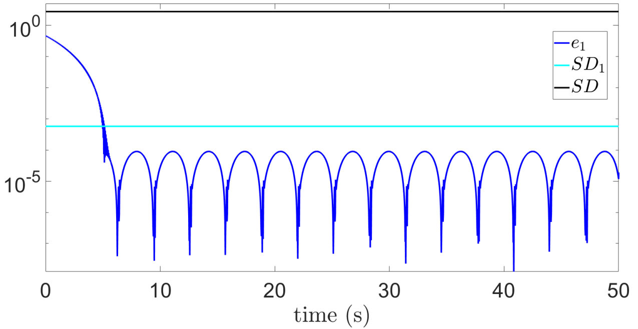
Semi-implicit homogeneous Euler differentiator



Semi-implicit differentiator State variable x_2 and estimated state variable $\hat{x}_2$ versus time (s)

Semi-implicit homogeneous Euler differentiator

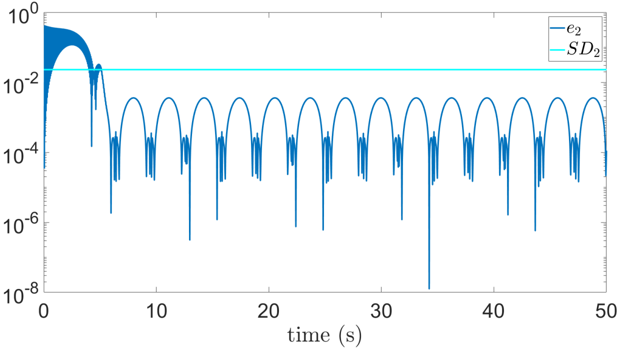
Numerical results - Semi-implicit differentiator



Semi-implicit differentiator Error $e_1 = \hat{x}_1 - x_1$ versus time (s) and related SD & SD_1 stability domains

Semi-implicit homogeneous Euler differentiator

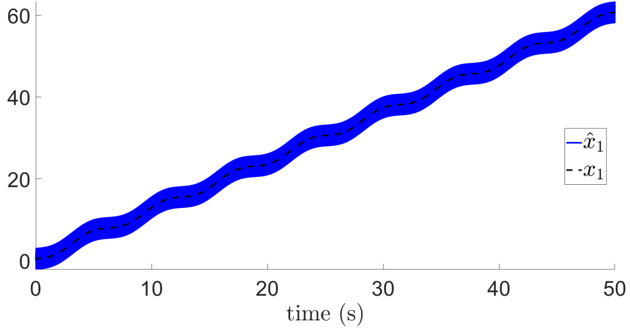
Numerical results - Semi-implicit differentiator



Semi-implicit differentiator Error $e_2 = \hat{x}_2 - x_2$ versus time (s) and related SD_2 stability domain

Semi-implicit homogeneous Euler differentiator

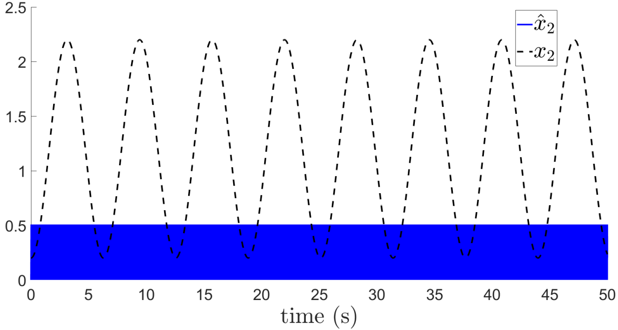
Numerical results - Explicit differentiator



Explicit differentiator State variable x_1 and estimated state variable $\hat{x}_1$ versus time (s)

Semi-implicit homogeneous Euler differentiator

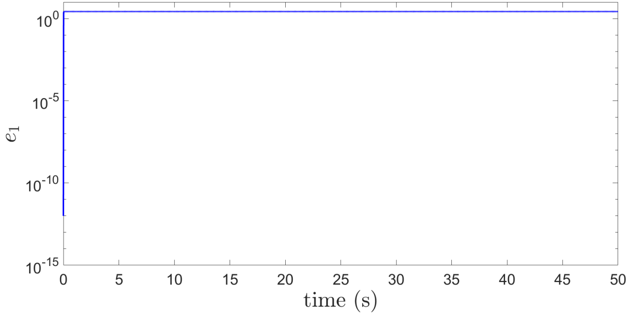
Numerical results - Explicit differentiator



Explicit differentiator State variable x_2 and estimated state variable $\hat{x}_2$ versus time (s)

Semi-implicit homogeneous Euler differentiator

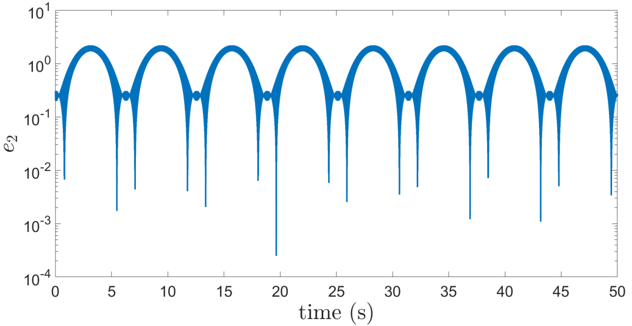
Numerical results - Explicit differentiator



Explicit differentiator Error $e_1 = \hat{x}_1 - x_1$ versus time (s)

Semi-implicit homogeneous Euler differentiator

Numerical results - Explicit differentiator



Explicit differentiator Error $e_2 = \hat{x}_2 - x_2$ versus time (s)

Semi-implicit homogeneous Euler differentiator

- The semi-implicit homogeneous Euler differentiator shows good estimations of states x_1 and x_2, they are not affected by the chattering phenomena even if the differentiator parameters λ_1 and λ_2 are oversized
- The estimation errors remain inside the range of the prescribed stability domains SD_1 and SD_2 as stated in Theorem 1
- Concerning the explicit homogeneous Euler differentiator, the reconstruction of the x_2 state fails for the same parameters λ_1 and λ_2 and the states are very affected by the chattering phenomena

Semi-implicit homogeneous Euler differentiator

- This paper proposes a semi-implicit Euler approximation of an homogeneous differentiator for a second-order system
- The main advantage of the proposed scheme is to keep the possibility of applying an implicit Euler approximation (combined with explicit one) when homogeneous differentiators are considered instead of classical sliding mode differentiators
- In that situation (homogeneous differentiators), the complete-implicit Euler approximation scheme fails and the complete explicit also for h sufficiently large (see the presentation of Subiksha)

Semi-Implicit Euler Discretization for Homogeneous Observer-based Control : one dimensional case

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Outline

- Introduction
- Problem statement
- Recall on Euler implicit sliding mode control
- Semi-implicit Euler discretization of homogeneous control and observer
- Semi-implicit Euler discretization of observer-based control
- Numerical results
- Conclusion

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Introduction

- Introduced by the work of Brogliato *et al.*, the *implicit discretization method* is well adapted to sliding-mode controllers and more generally to differential inclusion
- It aims to replace the sign function by an *implicit projector* with very promising results including
  - reduction of the chattering effect
  - robustness of the control under lower sampling frequencies
  - preservation of the global stability
- We propose in this work a semi-implicit based homogeneous controller to deal and cancel the effect of a class of perturbations

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Problem statement

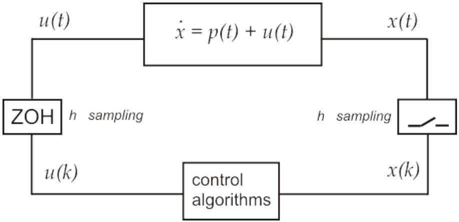
Towards implementation of discretized controllers

Consider a first order continuous perturbed system

$$\dot{x} = p(t) + u(t) \tag{1}$$

with $x \in R$ the state variable, $u \in R$ the control input and $p \in R$ the perturbation such that $|p(t)| < p_M$, p_M being a positive constant.

- Discretization towards software-in-the-loop implementation



Problem statement

Towards semi-implicit discretization

- From the standard homogeneous control sliding structure

$$u(t) = -\lambda\,|x(t)|^{\alpha}\mathrm{sgn}(x(t)) \tag{2}$$

we derive a *semi-implicit homogeneous control* in order to investigate the use of such "implicit "approaches for control and observation of perturbed systems

$\Rightarrow$ Reducing the chattering effect
$\Rightarrow$ Use of implicit method works for sliding-mode control and does not work for homogeneous control
$\Rightarrow$ Use of semi-implicit method for (2)

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Recall on Euler implicit sliding mode control

Principle of the implicit control from Acary & Brogliato

The exact discretized system considering $p = 0$, with a sampling-time h, is controlled by the *implicit projector* $\mathcal{N}_{\lambda,h}$ that gives

$$\begin{cases} x_{k+1} = x_k + h\, u_{k+1} \\ u_{k+1} = -\lambda\, \mathrm{sgn}(x_{k+1}) \end{cases} \tag{3}$$

where the $\mathrm{sgn}(x_{k+1})$ is evaluated thanks to the operator $\mathcal{N}_{\lambda,\,h}$ with $\lambda > 0$ that is defined as

$$\begin{cases} |x_k| < \lambda h \rightarrow \mathcal{N}_{\lambda,\,h} = \dfrac{x_k}{\lambda h} \qquad (\text{i.e. } x_{k+1} = 0) \\ |x_k| \geq \lambda h \rightarrow \mathcal{N}_{\lambda,\,h} = \mathrm{sgn}(x_k) \qquad (\text{i.e. } x_{k+1} \neq 0) \end{cases} \tag{4}$$

Recall on Euler implicit sliding mode control

Principle of the implicit control from Acary & Brogliato

Given the state variable x_k, the backward Euler implicit scheme

$$\begin{cases} x_{k+1} = x_k + h\, u_{k+1} \\ u_{k+1} = - \underbrace{\mathcal{N}_{\lambda,\, h}}_{\mathrm{sgn}(x_{k+1})} \end{cases}$$

- if $x_k \geq |\lambda h|$, then u_{k+1} belongs to the saturation mode [1] defined by $u_{k+1} = -\lambda\, \mathrm{sgn}(x_k)$,
- else u_{k+1} belongs to the linear mode and corresponds to a $1/\lambda$-contraction of $\frac{x_k}{\lambda h}$,

1. The $\mathrm{sgn}(x)$ function verifies : if $x > 0$, then $+1$; if $x < 0$ then -1 ; if $x = 0$ then $]-1, 1[$.

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The implicit-homogeneous based closed-loop reads

$$u_{k+1} = -\lambda \, |x_{k+1}|^{\alpha} \, \mathcal{N}_{\lambda,\, h} \tag{5}$$

If $|x_{k+1}|^{\alpha} = 0 \Rightarrow x_{k+1} = 0$, it is not possible to evaluate the projector

Semi-implicit Euler discretization of homogeneous control and observer

The homogeneous control based on semi-implicit Euler discretization u_{k+1}^{SI} is given by

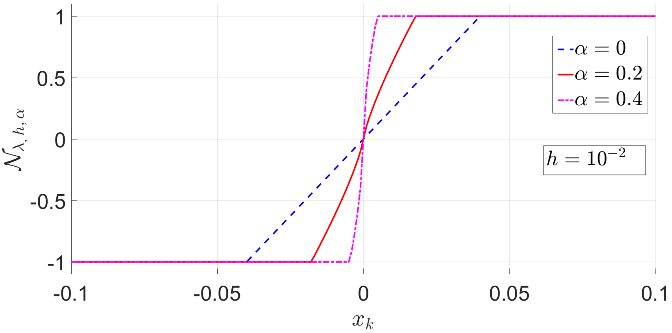
$$u_{k+1}^{SI} = -\lambda \left|x_k\right|^{\alpha} \mathcal{N}_{\lambda, h, \alpha}^{SI} \tag{6}$$

with

$$\mathcal{N}_{\lambda,h,\alpha}^{SI} := \begin{cases} \dfrac{|x_k|^{1-\alpha}}{\lambda h}\operatorname{sgn}(x_k) & \textbf{if} \quad |x_k|^{1-\alpha} < \lambda h \,(\textbf{i.e.}\, \tilde{x}_{k+1} = 0) \\ \operatorname{sgn}(x_k) & \textbf{if} \quad |x_k|^{1-\alpha} \geq \lambda h \,(\textbf{i.e.}\, \tilde{x}_{k+1} \neq 0) \end{cases} \tag{7}$$

where $\lambda > 0$ and $\alpha \in [0, 1[$ are constant parameters tuning ; the term $|x_k|^{\alpha}$ is the explicit part and the term $\mathcal{N}_{\lambda, h, \alpha}^{SI}$ constitutes the implicit part.

Semi-implicit Euler discretization of homogeneous control and observer



Examples of representation of $\mathcal{N}^{SI}_{\lambda,h,\alpha}$ versus α

Semi-implicit Euler discretization of homogeneous control and observer

Theorem 4 : For $h > 0$, the closed loop system, composed of the system $\dot{x} = p(t) + u(t)$ under the homogeneous control based on semi-implicit Euler discretization (6) action, reads as

$$x_{k+1} = x_k + h(p_{k+1} - \lambda \left|x_k\right|^{\alpha} \mathcal{N}^{SI}_{\lambda,\, h,\, \alpha}) \tag{8}$$

and converges in finite-time to 0 without perturbation (p_{k+1}), and converges in finite-time to hp_{k+1} in case of perturbation p_{k+1}. ■

Semi-implicit Euler discretization of homogeneous control and observer

The proposed semi-implicit observer reads as

$$\hat{x}_{k+1} = \hat{x}_k + h(\lambda_o |e_k|^{\alpha_o} \mathcal{N}^{SI}_{\lambda_o,\, h,\, \alpha_o} + u^{SI}_{k+1}) \tag{9}$$

where $\lambda_o > 0$ and $\alpha_o \in [0,\, 1[$ being constant tuning parameters. The projector aims to reconstruct the estimated state $\hat{x}$ from the error $e_k = x_k - \hat{x}_k$ including the perturbation.

Semi-implicit Euler discretization of homogeneous control and observer

Corollary 5 The estimation error e_{k+1} with the following dynamics converges in finite-time

$$e_{k+1} = e_k + h(p_k - \lambda_o |e_k|^{\alpha_o} \mathcal{N}^{SI}_{\lambda_o,\, h,\, \alpha_o}) \tag{10}$$

- to zero when system (1) is perturbation-free ($p = 0$) and exact discretization ;
- to hp_k when $p \neq 0$ and Euler discretization.

■

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Semi-implicit Euler discretization of homogeneous control and observer

The *semi-implicit discretized homogeneous observer-based* control reads

$$\left\{ \begin{array}{l} \hat{x}_{k+1} = \hat{x}_k + h(\lambda_o |e_k|^{\alpha_o} \mathcal{N}^{SI}_{\lambda_o,\, h,\, \alpha_o} + \bar{u}^{SI}_{k+1}) \\ \bar{u}^{SI}_{k+1} = -\lambda\, |x_k|^{\alpha} \mathcal{N}^{SI}_{\lambda,\, h,\, \alpha} + \lambda_o\, |e_k|^{\alpha_o} \mathcal{N}^{SI}_{\lambda_o,\, h,\, \alpha_o} \end{array} \right. \tag{11}$$

The observer-based control reads as a difference between the control projector $\mathcal{N}^{SI}_{\lambda,\, h,\, \alpha}$ and the observer projector $\mathcal{N}^{SI}_{\lambda_o,\, h,\, \alpha_o}$.

Theorem 6 : The closed loop system, composed of the system $\dot{x} = p(t) + u(t)$ controlled by the observer-based control (13), and for which the dynamics converges in a set bounded by $|h(p_{k+1} - p_k)|$. ∎

Semi-implicit Euler discretization of homogeneous control and observer

The *semi-implicit discretized homogeneous observer-based* control reads

$$\begin{cases} \hat{x}_{k+1} = \hat{x}_k + h(\lambda_o |e_k|^{\alpha_o} \mathcal{N}^{SI}_{\lambda_o,\, h,\, \alpha_o} + \bar{u}^{SI}_{k+1}) \\ \bar{u}^{SI}_{k+1} = -\lambda\, |x_k|^{\alpha} \mathcal{N}^{SI}_{\lambda,\, h,\, \alpha} + \lambda_o\, |e_k|^{\alpha_o} \mathcal{N}^{SI}_{\lambda_o,\, h,\, \alpha_o} \end{cases} \tag{12}$$

The observer-based control reads as a difference between the control projector $\mathcal{N}^{SI}_{\lambda,\, h,\, \alpha}$ and the observer projector $\mathcal{N}^{SI}_{\lambda_o,\, h,\, \alpha_o}$.

Theorem 6 : The closed loop system, composed of the system $\dot{x} = p(t) + u(t)$ controlled by the observer-based control (13), and for which the dynamics converges in a set bounded by $|h(p_{k+1} - p_k)|$. ■

Semi-implicit Euler discretization of homogeneous control and observer

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$$\begin{cases} \hat{x}_{k+1} = \hat{x}_k + h(\lambda_o |e_k|^{\alpha_o} \mathcal{N}^{SI}_{\lambda_o,\, h,\, \alpha_o} + \bar{u}^{SI}_{k+1}) \\ \bar{u}^{SI}_{k+1} = -\lambda \, |x_k|^{\alpha} \mathcal{N}^{SI}_{\lambda,\, h,\, \alpha} + \lambda_o \, |e_k|^{\alpha_o} \mathcal{N}^{SI}_{\lambda_o,\, h,\, \alpha_o} \end{cases} \quad (13)$$

The observer-based control reads as a difference between the control projector $\mathcal{N}^{SI}_{\lambda,\, h,\, \alpha}$ and the observer projector $\mathcal{N}^{SI}_{\lambda_o,\, h,\, \alpha_o}$.

Theorem 6 : The closed loop system, composed of the system $\dot{x} = p(t) + u(t)$ controlled by the observer-based control (13), and for which the dynamics converges in a set bounded by $|h(p_{k+1} - p_k)|$. ∎

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Consider the continuous system

$$\dot{x} = p(t) + u(t) \tag{14}$$

controlled by

$$\bar{u}_{k+1}^{SI} = -\lambda \left|x_k\right|^{\alpha} \mathcal{N}_{\lambda,\, h,\, \alpha}^{SI} + \lambda_o \left|e_k\right|^{\alpha_o} \mathcal{N}_{\lambda_o,\, h,\, \alpha_o}^{SI} \tag{15}$$

with $h = 10^{-3}$ s, and $x(0) = 0.45$, set also $\lambda = 1$ and $\lambda_o = 6$.

To ensure a faster dynamic of the observer than the control, consider $\lambda_o \gg \lambda$.

Numerical results

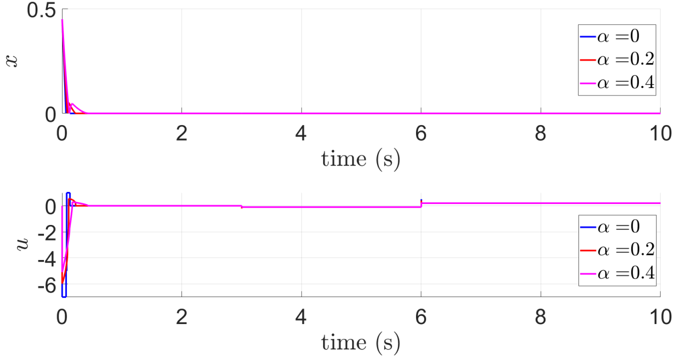
Control versus piecewise constant perturbation

Properties of observer-based explicit/semi-implicit controls are compared for different values for α and perturbation p defined as the following

$$\begin{array}{ll} 0 \leq t < 3, & p(t) = 0 \\ 3 \leq t < 6, & p(t) = 0.1 \\ 6 \leq t < 9, & p(t) = -0.2 \end{array}$$

Numerical results

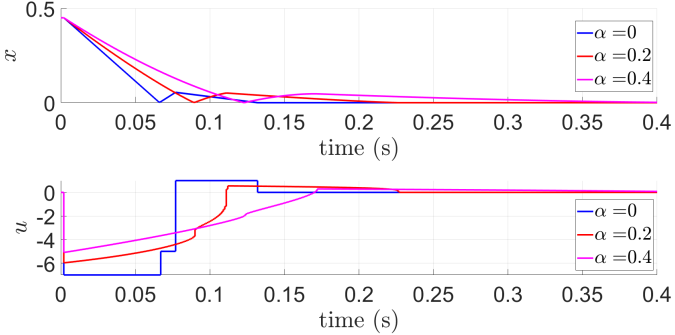
Control versus piecewise constant perturbation



Observer-based <u>semi-implicit</u> control - Piecewise perturbation State variable x (top) and control input u (bottom) versus time (s), for different values of α

Numerical results

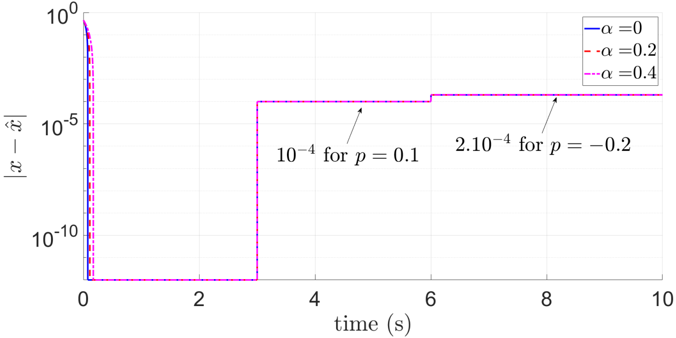
Control versus piecewise constant perturbation



Observer-based semi-implicit control - Piecewise perturbation - Focus on transient State variable x (top) and control input u (bottom) versus time (s), for different values of α

Numerical results

Control versus piecewise constant perturbation



Observer-based semi-implicit control - Piecewise perturbation
Estimation error $|x - \hat{x}|$ versus time (s), for different values of α

Numerical results

Control versus piecewise constant perturbation - Comparison of performances

α	Var_u	$\mathfrak{E}_u$	$\lvert\varepsilon\rvert_{p=0}$	$\lvert\varepsilon\rvert_{p=0.1}$	$\lvert\varepsilon\rvert_{p=-0.2}$
0	0	0.1602	$< 10^{-8}$	10^{-4}	2.10^{-4}
0.2	0	0.1602	$< 10^{-8}$	10^{-4}	2.10^{-4}
0.4	0	0.1602	$< 10^{-8}$	10^{-4}	2.10^{-4}

semi-implicit control

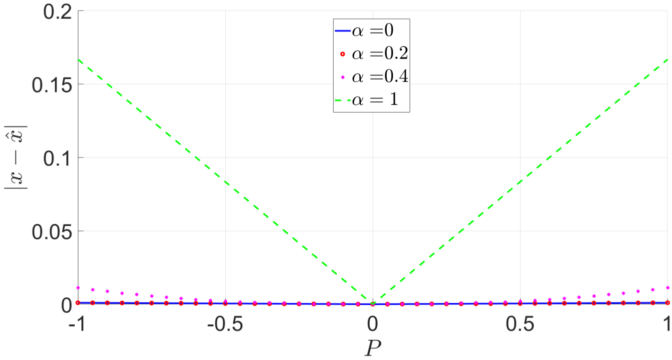
α	Var_u	$\mathfrak{E}_u$	$\lvert\varepsilon\rvert_{p=0}$	$\lvert\varepsilon\rvert_{p=0.1}$	$\lvert\varepsilon\rvert_{p=-0.2}$
0	3201	4	10^{-3}	10^{-3}	$1.2.10^{-3}$
0.2	0.2	0.16	$7.4.10^{-5}$	1.10^{-5}	$3.2.10^{-4}$
0.4	0.2	0.16	$3.1.10^{-6}$	$3.1.10^{-3}$	$1.7.10^{-2}$

explicit control

with $\varepsilon = |x - \hat{x}|$, $Var_u = \sum_i |u_{k+1} - u_k|$, $\mathfrak{E}_u = h \sum_k (u_k)^2$

Numerical results

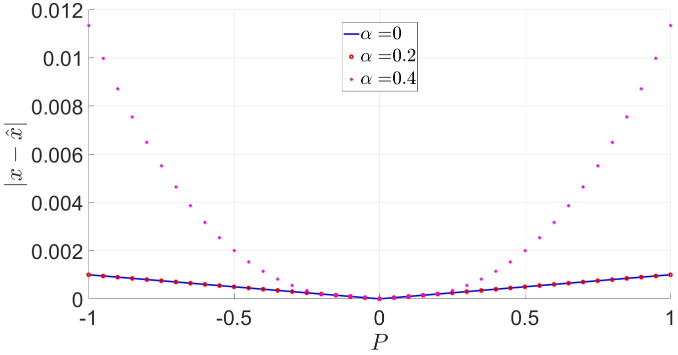
Control versus piecewise constant perturbation



Evaluation of the static error $|x - \hat{x}|$ according to the value of the perturbation P

Numerical results

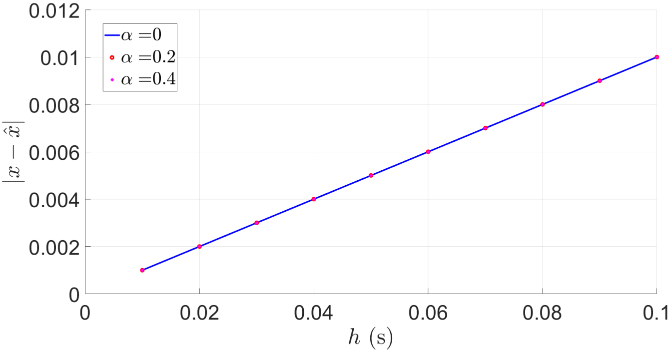
Control versus piecewise constant perturbation



Evaluation of the static error $|x - \hat{x}|$ according to the value of the perturbation P - focus

Numerical results

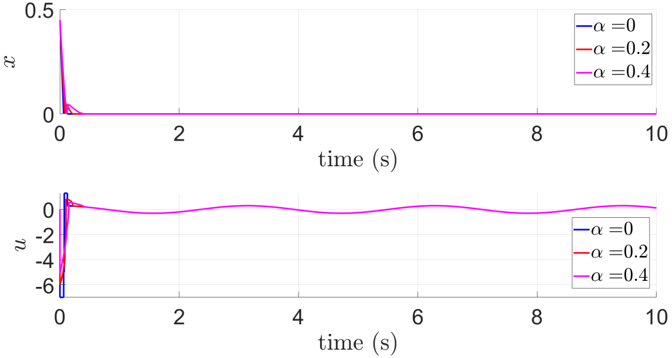
Control versus piecewise constant perturbation



Evaluation of the static error $|x - \hat{x}|$ according to the value of the sampling-time h

Numerical results

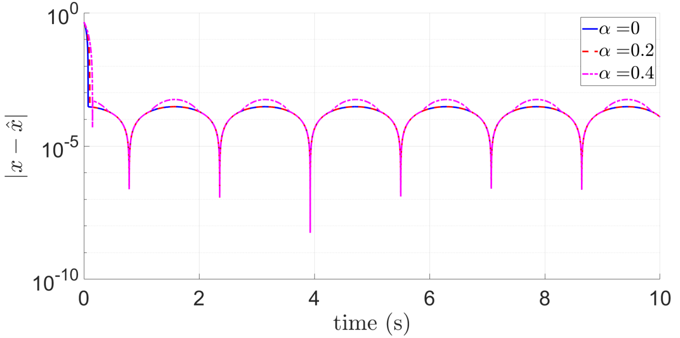
Control versus sine perturbation



Observer-based semi-implicit control - Sine perturbation. State variable x (top) and control input u (bottom) versus time (s), for different values of α

Numerical results

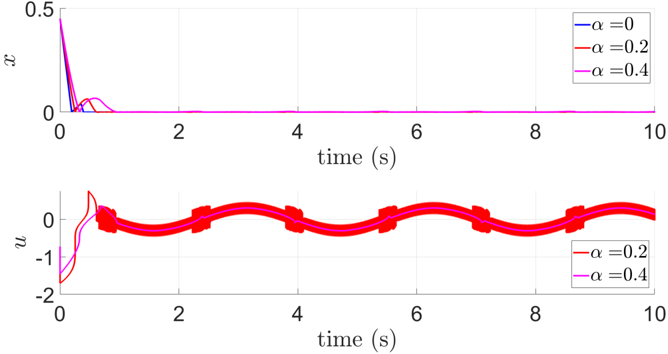
Control versus sine perturbation



Observer-based semi-implicit control - Sine perturbation.
Estimation error $|x-\hat{x}|$ versus time (s), for different values of α

Numerical results

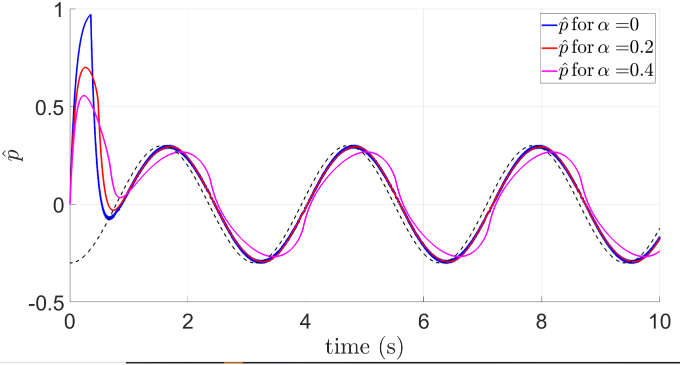
Control versus sine perturbation



Observer-based explicit control - Sine perturbation. State variable x (top) and control input u (bottom) versus time (s), for different values of α

Numerical results

Control versus sine perturbation



Observer-based explicit control - Sine perturbation. Estimated perturbation $\hat{p}$ versus time (s), for different values of α

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Conclusion

- This work has investigated the use of semi-implicit discretization approach for the control and observation of perturbed systems
- Homogeneous semi-implicit discretization has been introduced to control and observe perturbed systems
- Finally, an homogeneous observed-based semi-implicit control is proposed
- Future works include investigations of second order perturbed system as well as experimental validations on a pneumatic test-bed

Homogeneous Galerkin Method for Consistent Discretization of Infinite Dimensional Systems

Andrey Polyakov

Inria, Lille, France

10 September 2020

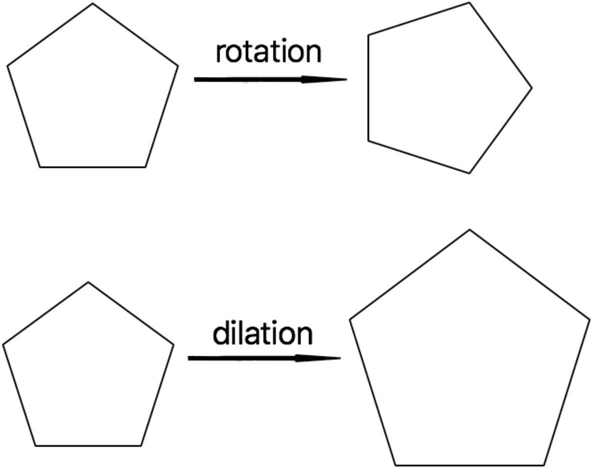
Annual Meeting of ANR DIGITSLID
LS2N, Nantes, France

Outline

I. Introduction

Homogeneity=Dilation Symmetry

Symmetry is an invariance with respect to a group of transformations.



Homogeneity is a dilation symmetry.

Generalized Homogeneity

Linearity = **Homogeneity** + **Additivity** + **Central Symmetry**

f is linear $\Leftrightarrow$ $f(e^s x) = e^s f(x)$ & $f(x+y) = f(x) + f(y)$ & $f(-x) = -f(x)$

Generalized Homogeneity

Linearity $=$ **Homogeneity** $+$ **Additivity** $+$ **Central Symmetry**

f is linear $\Leftrightarrow$ $f(e^s x) = e^s f(x)$ & $f(x+y) = f(x) + f(y)$ & $f(-x) = -f(x)$

Standard Homogeneity (*L. Euler, 18th century*):

$x \to e^s x$ (dilation) $\qquad$ $f(e^s x) = e^{\nu s} f(x)$ (symmetry)

$s \in \mathbb{R}$ - group parameter $\qquad$ $\nu \in \mathbb{R}$ - degree

Example: $x = (x_1, x_2)$, $f(x) = x_1 x_2 + x_2^2$

Generalized Homogeneity

Linearity = **Homogeneity** + **Additivity** + **Central Symmetry**

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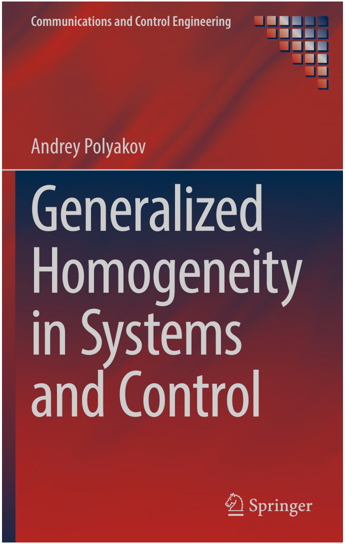
Generalized Homogeneity (*Zubov 1958, Khomenuk 1961, Hermes 1986, Kawski 1991, Coron & Praly 1991, Rosier 1992, Grune 2000, Levant 2003, Bhat & Bernstein 2005, Orlov 2005, Perruquetti & Moulay 2008, Andrieu et al 2008, ...*):

$x \to \mathbf{d}(s)x$ (dilation) $\qquad f(\mathbf{d}(s)x) = e^{\nu s} f(x)$, (symmetry)

Limit property: $\lim\limits_{s\to-\infty} \|\mathbf{d}(s)x\| = 0, \quad \lim\limits_{s\to+\infty} \|\mathbf{d}(s)x\| = +\infty, \quad \forall x \neq \mathbf{0}$

Example: $x = (x_1, x_2)$, $f(x) = x_1 + x_2^2$ with $\mathbf{d}(s) = \mathrm{diag}\{e^{2s}, e^s\}$


Communications and Control Engineering
Andrey Polyakov
Generalized Homogeneity in Systems and Control
Springer

Geometry-preserving approximations of evolution equations

Geometry-preserving approximations of evolution equations

Geometric Numerical Integration (ODE/PDE$\rightarrow$ Discrete-time):

- Finite-Difference Approximations preserving Lie Symmetries: *Dorodnitsyn 1989, Levi & Yamilov 1997, Heredero, Levi & Winternitz 2000, Bihlo & Valiquette 2017....*
- Symplectic integrators preserve some invariants of ODEs: *Channell & Scovel 1990, Leimkuhler & Reich 2004, Hairer, Wanner & Lubich 2006, ...*
- Energy preserving methods: *Quispel & McLaren 2008,...*
- Consistent discretization of ODEs (supported by ANR DIGITSLID): *Polyakov, Efimov & Brogliato 2019, Sanchez, Polyakov, Efimov 2020*

Geometry-preserving approximations of evolution equations

Geometric Numerical Integration (ODE/PDE$\to$ Discrete-time):

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Symmetry/Energy-preserving Galerkin methods (PDE $\to$ ODE):

- Reflection-symmetry-preserving projection: *Pla et al 2015*
- Energy preserving projection: *Liu & Xing 2016*
- Dilation-symmetry-preserving projection: *Polyakov 2020* (this paper)

II. Introduction to Homogeneous Evolution Equations

Linear dilations in Banach spaces $\mathbb{B}$

$\mathbb{B}$ - a real Banach space and $L(\mathbb{B},\mathbb{B})$ is a space of linear bounded operators

Definition

A one-parameter family $\mathbf{d}:\mathbb{R}\to\mathcal{L}(\mathbb{B},\mathbb{B})$ is said to be a **dilation** in $\mathbb{B}$ if

- **group property:** $\mathbf{d}(0)=I$, $\mathbf{d}(t+s)=\mathbf{d}(t)\mathbf{d}(s)$, $t,s\in\mathbb{R}$;
- **limit property:** $\lim\limits_{s\to-\infty}\|\mathbf{d}(s)u\|=0$ and $\lim\limits_{s\to+\infty}\|\mathbf{d}(s)u\|=\infty$ uniformly on $\mathcal{S}=\{u\in\mathbb{B}:\|u\|=1\}$.

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Definition

A dilation $\mathbf{d}$ in $\mathbb{B}$ is said to strongly (uniformly) continuous if the mapping $s\to\mathbf{d}(s)x$ (resp. $s\to\mathbf{d}(s)$) is continuous in $\mathbb{B}$ (resp. in $L(\mathbb{B},\mathbb{B})$) $\forall x\in\mathbb{B}$.

Example (Standard dilation)

The standard dilation $\mathbf{d}(s)=e^sI$ is uniformly continuous.

Generators of dilations

Definition (Generator of dilation)

A linear operator $G_{\mathbf{d}} : \mathcal{D}(G_{\mathbf{d}}) \subset \mathbb{B} \to \mathbb{B}$ defined as $G_{\mathbf{d}} x = \lim\limits_{s\to 0} \frac{\mathbf{d}(s)x - x}{s}$ on the domain $\mathcal{D}(G_{\mathbf{d}}) = \{x \in \mathbb{B} : \exists \lim\limits_{s\to 0} \frac{\mathbf{d}(s)x - x}{s}\}$ is called the **generator** of $\mathbf{d}$.

Theorem

If $\mathbf{d}$ is a strongly continuous dilation then its generator $G_{\mathbf{d}}$ is a linear closed densely defined operator and

$$\frac{d}{ds}\mathbf{d}(s)x = G_{\mathbf{d}}\mathbf{d}(s)x = \mathbf{d}(s)G_{\mathbf{d}}x, \quad \forall x \in \mathcal{D}(G_{\mathbf{d}}).$$

Linear Dilations in $\mathbb{R}^n$

Example

Any continuous linear **dilation** in $\mathbb{R}^n$ is a matrix-valued function given by

$$\mathbf{d}(s) = e^{sG_{\mathbf{d}}} = \sum_{i=0}^{+\infty} \frac{s^i G_{\mathbf{d}}^i}{i!}, \qquad s \in \mathbb{R},$$

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Standard dilation

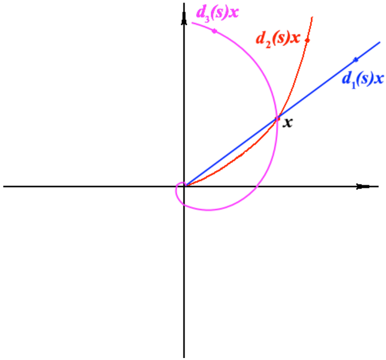
$\mathbf{d}_1(s) = e^s I, \quad G_\mathbf{d} = I \in \mathbb{R}^{n\times n}$

Weighted dilation

$\mathbf{d}_2(s) = \mathrm{diag}\{e^{r_i s}\}, G_\mathbf{d} = \mathrm{diag}\{r_i\} \succ 0$

Linear dilation

$\mathbf{d}_3(s) = e^{sG_d}$, $G_\mathbf{d}$ is anti-Hurwitz



Linear Dilations in Function Spaces

Example

Let us consider the one-parameter group of linear invertible operators in the Lebesgue space $L^p(\mathbb{R}^n, \mathbb{R}^m)$ given by

$$(\mathbf{d}(s)x)(z) = e^{\alpha s}x(e^{\beta s}z), \quad s \in \mathbb{R}, \quad x \in L^p(\mathbb{R}^n, \mathbb{R}^m), \quad z \in \mathbb{R}^n, \quad (1)$$

where $\alpha, \beta \in \mathbb{R}$ are constant parameters. Since

$$\|\mathbf{d}(s)x\|_{L^p} = e^{(\alpha - \beta n/p)s}\|x\|_{L^p}$$

then $\mathbf{d}$ is a dilation in $L^p(\mathbb{R}^n, \mathbb{R}^m)$ provided that $\alpha - \beta n/p > 0$.

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The generator of $\mathbf{d}$ in L^p is

$$(G_{\mathbf{d}}x)(z) = \alpha x(z) + \beta(z \cdot \nabla)x(z), \quad z \in \mathbb{R}^n, \quad x \in \mathcal{D}(G_{\mathbf{d}}) \subset L^p(\mathbb{R}^n, \mathbb{R}^m).$$

Monotone dilations

Definition (Monotone dilation)

The dilation $\mathbf{d}$ is strictly **monotone** if

$$\exists \gamma > 0 : \; \|\mathbf{d}(s)\| \leq e^{\gamma s}, \quad \forall s < 0,$$

where $\|\mathbf{d}(s)\| = \sup\{\|\mathbf{d}(s)x\| : \|x\| = 1\}$

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Let $\mathbb{H}$ be a real Hilbert space.

Theorem (Monotonicity in $\mathbb{H}$)

The dilation $\mathbf{d}$ is strictly monotone in $\mathbb{H}$ **if and only if** $\exists\gamma > 0$ and a set $\mathcal{D} \subset \mathcal{D}(G_{\mathbf{d}})$ in $\mathbb{H}$ such that

$$\langle G_{\mathbf{d}}x, x\rangle \geq \gamma\|x\|^2, \forall x \in \mathcal{D}.$$

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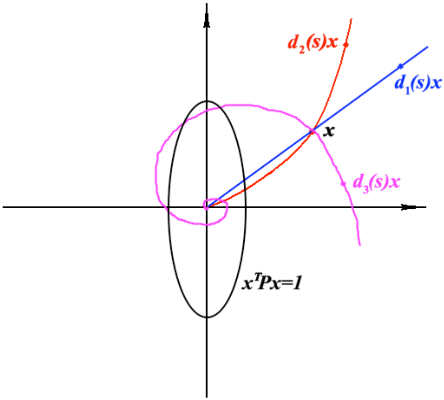

Proposition (Uniqueness of a homogeneous projection to the sphere)

If $\mathbf{d}$ is monotone then $\forall x \neq \mathbf{0}$ there exists a unique pair $(s_0, x_0) \in \mathbb{R} \times \mathcal{S}$ such that $x = \mathbf{d}(s_0)x_0$, where $\mathcal{S} = \{x : \|x\| = 1\}$ is the unit sphere.

Canonical Homogeneous Norm

Definition (a norm)

$p \in C(\mathbb{B}, \mathbb{R}_+)$ is a norm if

1) $p(x) = 0 \Leftrightarrow x = \mathbf{0}$
2) $p(\pm e^s x) = e^s p(x)$
3) $p(x + y) \leq p(x) + p(y)$

Definition (homogeneous "norm")

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$$\|x\|_{\mathbf{d}} = e^{s_x} \quad \text{where} \quad s_x \in \mathbb{R} : \|\mathbf{d}(-s_x)x\| = 1$$

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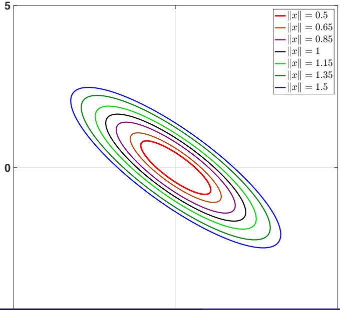
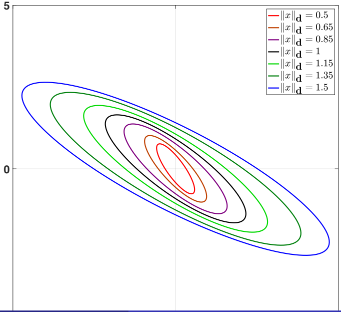




Homogeneous functions and vector fields

Definition (Homogeneous functional)

A possibly unbounded **functional** $h : \mathcal{D}(h) \subset \mathbb{B} \to \mathbb{R}$ is **d**-homogeneous of the degree $\mu \in \mathbb{R}$ if $\mathbf{d}(s)\mathcal{D}(h) \subset \mathcal{D}(h)$ and

$$h(\mathbf{d}(s)x) = e^{\mu s} h(x), \quad \forall x \in \mathcal{D}(h), \quad \forall s \in \mathbb{R}.$$

Definition (Homogeneous Operator)

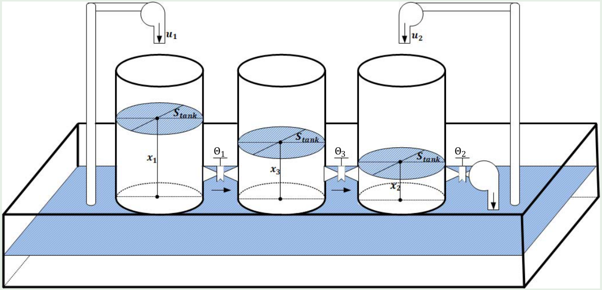
A possibly unbounded **operator** $f : \mathcal{D}(f) \subset \mathbb{B} \to \mathbb{B}$ is **d**-homogeneous of the degree $\mu \in \mathbb{R}$ if $\mathbf{d}(s)\mathcal{D}(f) \subset \mathcal{D}(f)$ and

$$f(\mathbf{d}(s)x) = e^{\mu s}\mathbf{d}(s) f(x), \quad \forall x \in \mathcal{D}(f), \quad \forall s \in \mathbb{R}.$$

Example (Three-tank system)

$$
\begin{aligned}
\dot{x}_1 &= S_{tank}^{-1}\left[-\theta_1 \lceil x_1 - x_3 \rfloor^{0.5} + u_1\right], \\
\dot{x}_2 &= S_{tank}^{-1}\left[\theta_3 \lceil x_3 - x_2 \rfloor^{0.5} - \theta_2 \lceil x_2 \rfloor^{0.5} + u_2\right], \\
\dot{x}_3 &= S_{tank}^{-1}\left[\theta_1 \lceil x_1 - x_3 \rfloor^{0.5} - \theta_3 \lceil x_3 - x_2 \rfloor^{0.5}\right],
\end{aligned}
\qquad (2)
$$

where $\lceil \rho \rfloor^{0.5} = |\rho| \operatorname{sign}(\rho)$.

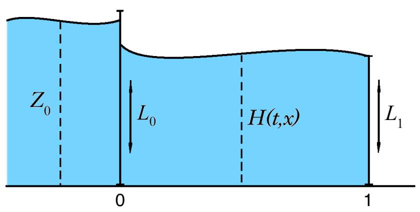


The model of the three-tank system is standard homogeneous $\mathbf{d}(s) = e^s I_3$ of the degree -0.5 for $u_1 = 0, u_2 = 0$.

Example (A flow in open channels – Saint-Venant Equation)

$$\frac{\partial H}{\partial t} = -\frac{\partial}{\partial x}(HV),$$
$$\frac{\partial V}{\partial t} = -\frac{\partial}{\partial x}\left(\tfrac{1}{2}V^2 + gH\right),$$
$$H(t,0)V(t,0) - (Z_0 - L_0)^{3/2} = 0,$$
$$H(t,1)V(t,1) - (H(t,1) - L_1)^{3/2} = 0,$$

where H is the water level and V is the water velocity.



Let $f : \mathcal{D}(f) \subset \mathbb{B} \to \mathbb{B} := \mathbb{C}([0,1],\mathbb{R}) \times \mathbb{C}([0,1],\mathbb{R})$ be defined on the domain

$$\mathcal{D}(f) = \left\{ (H,V) \in \mathbb{C}^1([0,1],\mathbb{R}_+) \times \mathbb{C}^1([0,1],\mathbb{R}) : \begin{array}{c} H(0)V_2(0) = 0; \\ H(1)V(1) = H^{3/2}(1) \end{array} \right\}$$

as follows

$$f(H,V) = \begin{pmatrix} -\frac{\partial}{\partial x} HV) \\ -\frac{\partial}{\partial x}\left(gH + \frac{1}{2}V^2\right), \quad (H,V) \in \mathcal{D}(f) \end{pmatrix},$$

The operator f is $\mathbf{d}$-homogeneous of degree $\mu = 1$ with respect to the weighted dilation $\mathbf{d}(s)(H,V) = (e^{2s}H, e^{s}V)$,

$$f \circ \mathbf{d}(s) = e^{s}\mathbf{d}(s) \circ f.$$

Example (Laplace operator)

$$\Delta = \frac{\partial^2}{\partial z_1^2} + ... + \frac{\partial^2}{\partial z_n} : \mathcal{D}(\Delta) \subset L^2(\mathbb{R}^n, \mathbb{R}^m) \to L^2(\mathbb{R}^n, \mathbb{R}^m)$$

is **d**-homogeneous of the degree 2β with respect to the dilation

$$(\mathbf{d}(s)x)(z) = e^{\alpha s}x(e^{\beta s}z), \quad x \in L^2, \quad z = (z_1, ..., z_n)^\top \in \mathbb{R}^n, \quad \alpha > n\beta/2.$$

$$(\Delta\mathbf{d}(s)x)(z) = \Delta e^{\alpha s}x(e^{\beta s}z) = e^{2\beta s}(\mathbf{d}(s)\Delta x)(z).$$

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Example (Navier–Stokes equations)

The classical model of the flow of an incompressible viscous fluid is

$$\begin{aligned} \frac{\partial u}{\partial t} &= \nu \Delta u - (u \cdot \nabla) u - \nabla p, \\ \mathbf{0} &= \mathrm{div} u \end{aligned} \qquad (3)$$

where u denotes the velocity of a fluid in $\mathbb{R}^3$, p denotes the scalar pressure and $\nu > 0$ denotes viscosity of the fluid. It is **d**-homogeneous as well.

Homogeneous evolution equations

Let us consider the nonlinear evolution equation in a Banach space

$$\dot{x} = Ax + f(x), \quad t > 0, \quad x(0) = x_0 \tag{4}$$

where $x(t), x_0 \in \mathbb{B}$, $\dot{x}(t) = \lim_{h\to 0} \frac{x(t+h)-x(t)}{h}$, $A : \mathcal{D}(A) \subset \mathbb{B} \to \mathbb{B}$ and $f : \mathcal{D}(f) \subset \mathbb{B} \to \mathbb{B}$ are linear and, respectively, a non-linear (possibly unbounded) closed densely defined operators, $\mathcal{D}(A) \subset \mathcal{D}(f)$.

Theorem

Let A and f be $\mathbf{d}$-homogeneous operators of a degree $\mu \in \mathbb{R}$. If $x : [0, T) \to \mathbb{B}$ is a solution of (4) then for any $s \in \mathbb{R}$ the function $x^s : [0, e^{-\mu s} T) \to \mathbb{B}$ given by $x^s(t) := \mathbf{d}(s) x(e^{\mu s} t)$, $t \in [0, e^{-\mu s} T)$ is a solution of the evolution equation (4) as well.

Notice, if $\mathbb{B} = \mathbb{H}$ the equation (4) admits the equivalent *week formulation*

$$\langle \dot{x}, v\rangle = \langle Ax + f(x), v\rangle, \quad \forall v \in V, t > 0$$

where $V \subset \mathbb{H}$ is a linear subspace dense in $\mathbb{H}$.

III. Homogeneous Galerkin Method

Classical Galerkin method (for linear functional equations)

Strong formulation

Find $x \in \mathcal{D}(A)$ such that $Ax = y$,

where $A : \mathcal{D}(A) \subset \mathbb{H} \to \mathbb{H}$ - linear operator and $y \in \mathbb{H}$

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Weak formulation

Find $x_v \in V$ such that $\langle y, v \rangle = \langle Ax_v, v \rangle, \quad \forall v \in V$,
where $V \subset \mathcal{D}(A)$ is a linear subspace dense in $\mathbb{H}$.
Example: $A = \frac{\partial^2}{\partial z^2}, \mathbb{H} = L^2(\mathbb{R}, \mathbb{R}), \mathcal{D}(A) = H^2(\mathbb{R}, \mathbb{R}), V = C_c^\infty(\mathbb{R}, \mathbb{R})$

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Galerkin projection

$$\textbf{Find}\quad x_v \in V \text{ such that } \langle y, v\rangle = \langle Ax_v, v\rangle, \quad \forall v \in V,$$

where $V = \text{span}\{h_1, h_2, ..., h_n\}$ and $\{h_i\}_{i=1}^n \in \mathbb{H}$ is an orthonormal family.

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If $x_v = \sum_{i=1}^n \tilde{z}_i h_i, y = \sum_{i=1}^n \tilde{y}_i h_i, v = \sum_{i=1}^n \tilde{v}_i h_i, \tilde{x}, \tilde{y}, \tilde{v} \in \mathbb{R}^n$ then

$\langle y, v \rangle = \langle Ax_v, v \rangle, \forall v \in V \quad \Leftrightarrow \quad \tilde{v}^\top \tilde{y} = \tilde{v}^\top A_n \tilde{x}, \forall v \in \mathbb{R}^n \quad \Leftrightarrow \quad \tilde{y} = A_n \tilde{x}$

where $A_n \in \mathbb{R}^{n \times n}$, $(A_n)_{i,j} = \langle Ah_j, h_i \rangle$.

Galerkin approximation of a dilation $\mathbf{d}$

If $x_0 \in \mathcal{D}(G_{\mathbf{d}}) \subset \mathbb{H}$ then $x(s) = \mathbf{d}(s)x_0$ fulfills $\dot{x}(s) = G_{\mathbf{d}}x(s),\, s \in \mathbb{R},\, x(0) = y$

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Galerkin projection of the dilation

Find $x_v \in C(\mathbb{R}, V)$ such that $\begin{cases} \langle \dot{x}_v(s), v\rangle = \langle G_{\mathbf{d}} x_v(t), v\rangle, & \forall s \in \mathbb{R}, \\ \langle x(0), v\rangle = \langle x_0, v\rangle, & \forall v \in V \end{cases}$,

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If $\Pi_n : \mathbb{H} \to \mathbb{R}^n$ is defined as $\Pi_n x = \begin{pmatrix} \langle x, h_1\rangle \\ \langle x, h_2\rangle \\ \ldots \\ \langle x, h_n\rangle \end{pmatrix}$ then Galerkin method gives

$$\frac{d}{ds}\tilde{x}(s) = G_{\mathbf{d}_n}\tilde{x}(s), \quad s \in \mathbb{R}, \quad \tilde{x}(0) = \Pi_n y \in \mathbb{R}^n,$$

where $x_v(t) = \sum_{i=1}^n \tilde{x}_i(t) h_i$, $\tilde{x} \in \mathbb{R}^n$ and $G_{\mathbf{d}_n} \in \mathbb{R}^{n\times n}$ is the Galerkin projection of $G_{\mathbf{d}}$, i.e. $(G_{\mathbf{d}_n})_{i,j} = \langle G_{\mathbf{d}} h_j, h_i\rangle$.

$$\mathbf{d}_n(s) = e^{sG_{\mathbf{d}_n}}, s \in \mathbb{R} \; - \; \text{Galerkin projection of } \mathbf{d}$$

Notice if $\langle G_{\mathbf{d}} y, y\rangle \geq \gamma \|y\|^2$ then $G_{\mathbf{d}_n} + G_{\mathbf{d}_n}^\top \succ \gamma I_n$.

Example

$$(\mathbf{d}(s)x)(z) = e^{\alpha s}x(e^{\beta s}z), \quad z, s \in \mathbb{R}, \quad x \in L^2(\mathbb{R}, \mathbb{R}), \alpha > \beta/2.$$

Let us consider the Hermite functions

$$h_i(z) = \frac{(-1)^{i-1}}{\sqrt{2^{i-1}(i-1)!\sqrt{\pi}}} e^{\frac{z^2}{2}} \frac{d^{i-1}}{dy^{i-1}} e^{-z^2}, \quad z \in \mathbb{R}, i = 1, 2, \ldots \tag{5}$$

The finite-dimensional projection of $G_{\mathbf{d}}$ is

$$G_{\mathbf{d}_n} = \begin{pmatrix} \frac{2\alpha-\beta}{2} & 0 & \beta\sqrt{\frac{1}{2}} & 0 & 0 & 0 & \ldots \\ 0 & \frac{2\alpha-\beta}{2} & 0 & \beta\sqrt{\frac{3}{2}} & 0 & 0 & \ldots \\ -\beta\sqrt{\frac{1}{2}} & 0 & \frac{2\alpha-\beta}{2} & 0 & \beta\sqrt{3} & 0 & \ldots \\ 0 & -\beta\sqrt{\frac{3}{2}} & 0 & \frac{2\alpha-\beta}{2} & 0 & \beta\sqrt{5} & \ldots \\ 0 & 0 & -\beta\sqrt{3} & 0 & \frac{2\alpha-\beta}{2} & 0 & \ldots \\ 0 & 0 & 0 & -\beta\sqrt{5} & 0 & \frac{2\alpha-\beta}{2} & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix} \tag{6}$$

and the finite-dimensional projection of the dilation $\mathbf{d}$ is given by

$$\mathbf{d}_n(s) = e^{sG_{\mathbf{d}_n}} = e^{(\alpha-0.5\beta)s}e^{s\Xi}, \quad s \in \mathbb{R}, \quad \Xi = -\Xi^\top = G_{\mathbf{d}_n} - (\alpha - 0.5\beta)I_n.$$

The latter means that $\|\tilde{x}\|_{\mathbf{d}_n} = \|\tilde{x}\|_{\mathbb{R}^n}^{\frac{1}{\alpha-0.5\beta}}$ for any $\tilde{x} \in \mathbb{R}^n$.

Homogeneous Polar coordinates

If A and f are homogeneous operators of a degree $\nu \in \mathbb{R}$ then denoting

$$z(t) = \mathbf{d}(-\ln \|x(t)\|_{\mathbf{d}})x(t), \quad r(t) = \|x(t)\|_{\mathbf{d}}$$

(homogeneous polar coordinates[1])

[1] Homogeneous polar cooridnates in $\mathbb{R}^n$ were introduced by Laurent Praly, CDC, 1997

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(homogeneous polar coordinates[1])

the evolution equation (4) recalled here as

$$\dot{x}(t) = Ax(t) + f(x(t)), \quad t \in (0, T), \quad x(0) = x_0$$

can be rewritten as follows

$$\begin{cases} r^{-\nu}(t)\mathbf{d}(-\ln r(t))\frac{d(\mathbf{d}(\ln r(t))z(t))}{dt} = Az(t) + f(z(t)), \quad t > 0, \\ z(0) = \mathbf{d}(-\ln \|x_0\|_{\mathbf{d}})x_0. \end{cases} \tag{7}$$

and

$$\dot{r}(t) = r^{\nu+1}(t)\frac{\langle Az(t)+f(z(t)),z(t)\rangle}{\langle G_{\mathbf{d}}z(t),z(t)\rangle}, \quad t > 0, \quad r(0) = \|x_0\|_{\mathbf{d}}. \tag{8}$$

[1] Homogeneous polar cooridnates in $\mathbb{R}^n$ were introduced by Laurent Praly, CDC, 1997

Homogeneous Galerkin Projection

Galerkin projection

$$\begin{gathered}\textbf{find } \phi_v \in C([0,T),V) \textbf{ and } \tilde r \in C([0,T),\mathbb{R}_+) \textbf{ such that}\\ \left\langle \tilde r^{-\nu}(t)\mathbf{d}(-\ln \tilde r(t))\tfrac{d}{dt}(\mathbf{d}(\ln \tilde r(t))\phi_v(t))-A\phi_v(t)-f(\phi_v(t)),v\right\rangle \overset{a.e.}{=} 0,\\ \tfrac{d}{dt}\tilde r(t) \overset{a.e.}{=} \tilde r^{\nu+1}(t)\tfrac{\langle A\phi_v(t)+f(\phi_v(t)),\phi_v(t)\rangle}{\langle G_{\mathbf{d}}\phi_v(t),\phi_v(t)\rangle},\\ \langle \phi_v(0),v\rangle = \langle \mathbf{d}(-\ln\|x_0\|_{\mathbf{d}})x_0,v\rangle, \quad \text{and} \quad \tilde r(0)=\|x_0\|_{\mathbf{d}},\\ \forall v\in V, \forall t\in(0,T),\end{gathered} \tag{9}$$

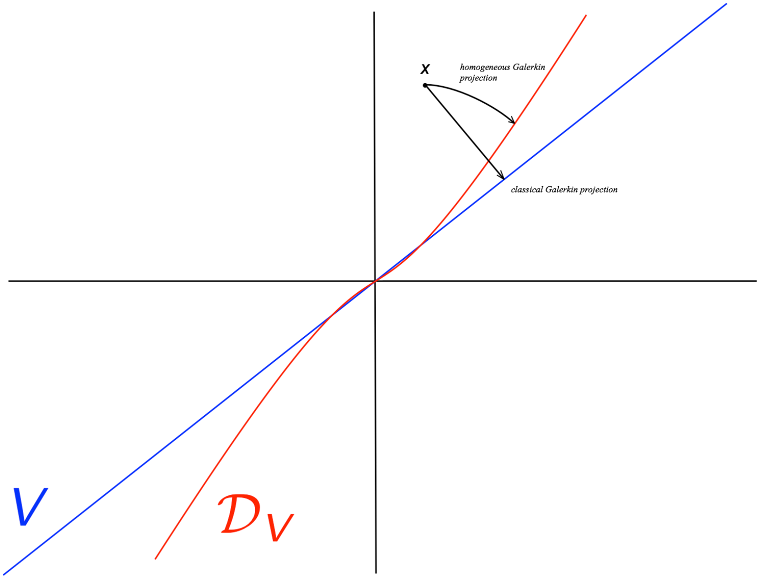
where $V \subset \mathbb{H}$ is a linear subspace of $\mathbb{H}$ and $S_V = \{z \in V : \|z\|_{\mathbb{H}} = 1\}$ is the unit sphere in V.

$$x_v(t) = \mathbf{d}(\ln \tilde r(t))\phi_v(t), \quad t \in [0,T) \tag{10}$$

is a Galerkin-like projection of the strong solution $x : [0,T) \to \mathbb{H}$ of the system (4) on the **d**-homogeneous cone

$$\mathcal{D}_V := \bigcup_{s\in\mathbb{R}} \mathbf{d}(s)S_V. \tag{11}$$

Geometric illustration of homogeneous projection



On existence of a homogeneous Galerkin projection

Theorem

Let the operators A, f be $\mathbf{d}$-homogeneous of the degree $\nu \in \mathbb{R}$ and satisfy certain regularity assumptions, $\mathcal{D}(A) \subset \mathcal{D}(f)$, and $h_i \in \mathcal{D}(A) \cap \mathcal{D}(G_{\mathbf{d}})$, $i = 1, 2, ..., n$ be an orthonormal basis in $V = \mathrm{span}\{h_1, ..., h_n\}$. Then for any $x_0 \in \mathcal{D}_V$ there exists a pair $\phi_v, \tilde{r}$ satisfying (9) *such that*

$$\phi_v(t) = \sum_{i=1}^{n} \tilde{\phi}_i(t) h_i, \quad t \in [0, T),$$

where the pair $\tilde{\phi}(t) = (\tilde{\phi}_1(t), ..., \tilde{\phi}_n(t))^\top \in S_{\mathbb{R}^n}$, $\tilde{r}(t) \in \mathbb{R}_+$ is the unique classical solution of the following ODE

$$\begin{cases} \frac{d\tilde{\phi}(t)}{dt} = \tilde{r}^{\nu}(t)\left(A_n\tilde{\phi}(t) + \tilde{f}(\tilde{\phi}(t))\right) - \tilde{r}^{\nu}(t)\frac{\tilde{\phi}^\top(t)A_n\tilde{\phi}(t) + \tilde{\phi}^\top(t)\tilde{f}(\tilde{\phi}(t))}{\tilde{\phi}^\top(t)G_{\mathbf{d}_n}\tilde{\phi}(t)} G_{\mathbf{d}_n}\tilde{\phi}(t), \\ \frac{d\tilde{r}(t)}{dt} = \tilde{r}^{\nu+1}(t)\frac{\tilde{\phi}^\top(t)A_n\tilde{\phi}(t) + \tilde{\phi}^\top \tilde{f}(\tilde{\phi}(t)}{\tilde{\phi}^\top(t)G_{\mathbf{d}_n}\tilde{\phi}(t)}, \quad t \in (0, T), \\ \tilde{\phi}(0) = \Pi_n \mathbf{d}(-\ln \|x_0\|_{\mathbf{d}})x_0, \quad \tilde{r}(0) = \|x_0\|_{\mathbf{d}}, \end{cases} \tag{12}$$

where $G_{\mathbf{d}_n}$ and A_n are Galerkin projections of $G_{\mathbf{d}}$ and A, respectively.

Properties of homogenenous Galerkin Projection

- Introducing

$$\tilde{x}(t) = \mathbf{d}_n(\ln \tilde{r}(t))\tilde{\phi}(t)$$

we derive(12) is homeomorphic on $\mathbb{R}^n$ and diffeomorphic on $\mathbb{R}^n\backslash\{\mathbf{0}\}$

$$\frac{d\tilde{x}(t)}{dt} = \|\tilde{x}(t)\|_{\mathbf{d}_n}^{\nu}\mathbf{d}_n(\ln\|\tilde{x}(t)\|_{\mathbf{d}})\Big(\tilde{A}_n\mathbf{d}_n(-\ln\|\tilde{x}\|_{\mathbf{d}_n})\tilde{x}(t)+\tilde{f}(\mathbf{d}_n(-\ln\|\tilde{x}(t)\|_{\mathbf{d}_n})\tilde{x}(t))\Big),$$
$$\tilde{x}(0) = \mathbf{d}_n(\ln\|x_0\|_{\mathbf{d}})\Pi_n\mathbf{d}(-\ln\|x_0\|_{\mathbf{d}})x_0, \quad t\in(0,T). \tag{13}$$

- If $x_v \in C([0, T), \mathcal{D}_V)$ be a solution of (9) for $x_0 \in \mathcal{D}_V$ then $\forall s \in \mathbb{R}^n$ the function $x_v^s \in C([0, e^{-\nu s}T), \mathcal{D}_V)$ given by

$$x_v^s(t) = \mathbf{d}(s)x_v(e^{\nu s})$$

is the solution of (9) with the scaled initial condition $x(0) = \mathbf{d}(s)x_0$.

- The obtained finite-dimensional projection of the nonlinear evolution equation (4) preserve stability properties of the original system and the convergence rates (finite-time/fixed-time stability).

Consider the Burgers equation

$$\frac{\partial x}{\partial t} = \frac{\partial^2 x}{\partial z^2} - x\frac{\partial x}{\partial z}, \quad t > 0, \quad x(0,z) = x_0(z), \quad z \in \mathbb{R}, \quad x_0 \in L^2 \quad (14)$$

which has the exact solution

$$[x(t)](z) = -2\frac{\partial}{\partial z}\ln\left\{\frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{+\infty} e^{-\frac{(z-\sigma)^2}{4t}-\frac{1}{2}\int_0^\sigma x_0(s)ds} d\sigma\right\} \quad (15)$$

Compare the classical and homogeneous Galerkin projections for $n = 5$, the Hermite basis

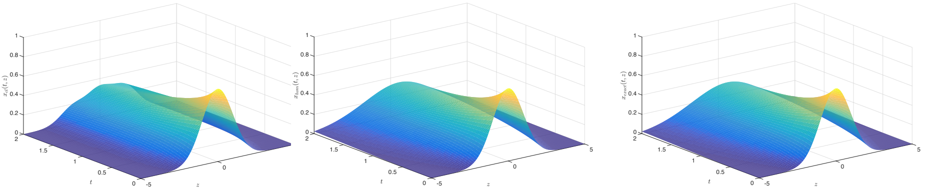
$$h_i(z) = \frac{(-1)^{i-1}}{\sqrt{2^{i-1}(i-1)!\sqrt{\pi}}} e^{\frac{z^2}{2}} \frac{d^{i-1}}{dy^{i-1}} e^{-z^2}, \quad z \in \mathbb{R}, i = 1, 2, ...$$

and two initial conditions

$$x_0 = h_1 \in V \qquad \text{or} \qquad x_0 = \xi \notin V,$$

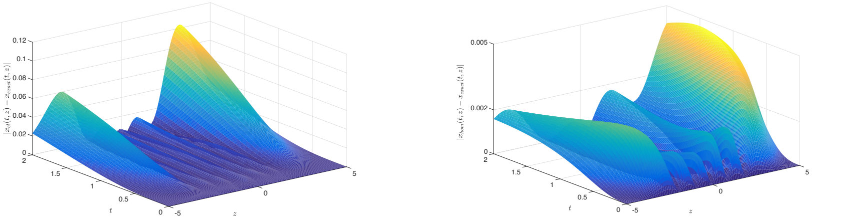
where

$$\xi(z) = \begin{cases} 1 & \text{if} \quad |z| \le 1, \\ 0 & \text{if} \quad |z| > 1, \end{cases} \qquad z \in \mathbb{R}.$$



(a) Classical Galerkin (b) Homogeneous Galerkin (c) Exact solution

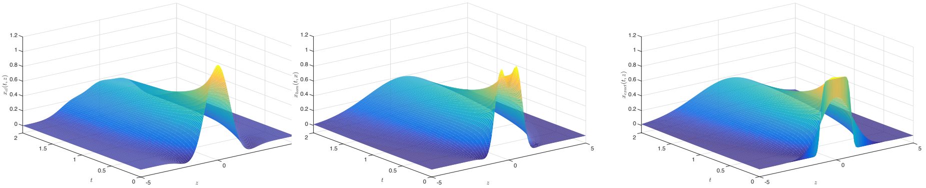
Figure: Approximate and exact solutions for $x_0 = h_1$



(a) The classical Galerkin method (b) The homogeneous Galerkin method

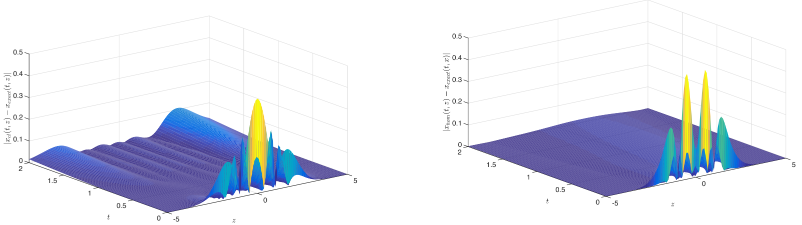
Figure: Approximation errors for $x_0 = h_1$

Simulation results $x_0 = \xi$



(a) Classical Galerkin (b) Homogeneous Galerkin (c) Exact solution

Figure: Approximate and exact solutions for $x_0 = \xi$



(a) The classical Galerkin method (b) The homogeneous Galerkin method

Figure: Approximation errors for $x_0 = \xi$

Summary

- A homogeneous Galerkin method is proposed for homogeneous evolution equations in Hilbert spaces.
- It preserves
  - the homogeneity(dilation symmetry) in the finite-dimensional projection;
  - the convergence rates (finite-time and fixed-time stability) of the original system.
- Simulations shows a large improvement of the approximation precision for small number of basis functions.

Consistent Discretization of Homogeneous System Using Lyapunov Function

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Inria, Lille, France

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Annual Meeting of ANR DIGITSLID
LS2N, Nantes, France

Outline

1. Introduction
   - Homogeneity
   - Geometry-preserving approximations of evolution equations
   - The problem of consistent discretization
2. Consistent Discretization using Lyapunov Function
   - Lyapunov function of homogeneous system
   - Polar coordinates for stable homogeneous system
3. Examples
   - Consistent discretization of quasi-continuous 2-SM controller
   - Consistent discretization of a positive degree system

I. Introduction

Generalized Homogeneity

Linearity $=$ **Homogeneity** $+$ **Additivity** $+$ **Central Symmetry**

f is linear $\Leftrightarrow f(e^s x) = e^s f(x)$ & $f(x+y) = f(x) + f(y)$ & $f(-x) = -f(x)$

Generalized Homogeneity

Linearity = **Homogeneity** + **Additivity** + **Central Symmetry**

f is linear $\Leftrightarrow$ $f(e^s x) = e^s f(x)$ & $f(x+y) = f(x) + f(y)$ & $f(-x) = -f(x)$

Standard Homogeneity (*L. Euler, 18th century*):

$x \to e^s x$ (dilation) $\qquad f(e^s x) = e^{\nu s} f(x)$ (symmetry)

$s \in \mathbb{R}$ - group parameter $\qquad \nu \in \mathbb{R}$ - degree

Example: $x = (x_1, x_2)$, $f(x) = x_1 x_2 + x_2^2$

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Generalized Homogeneity (*Zubov 1958, Khomenuk 1961, Hermes 1986, Kawski 1991, Coron & Praly 1991, Rosier 1992, Grune 2000, Levant 2003, Bhat & Bernstein 2005, Orlov 2005, Perruquetti & Moulay 2008, Andrieu et al 2008, ...*):

$x \to \mathrm{d}(s)x$ (dilation) $\qquad f(\mathrm{d}(s)x) = e^{\nu s} f(x)$, (symmetry)

Limit property: $\lim_{s \to -\infty} \|\mathrm{d}(s)x\| = 0, \quad \lim_{s \to +\infty} \|\mathrm{d}(s)x\| = +\infty, \quad \forall x \neq 0$

Example: $x = (x_1, x_2)$, $f(x) = x_1 + x_2^2$ with $\mathrm{d}(s) = \mathrm{diag}\{e^{2s}, e^s\}$

Linear Dilations in $\mathbb{R}^n$

Example

Any continuous linear **dilation** in $\mathbb{R}^n$ is a matrix-valued function given by

$$\mathrm{d}(s) = e^{sG_{\mathrm{d}}} = \sum_{i=0}^{+\infty} \frac{s^i G_{\mathrm{d}}^i}{i!}, \qquad s \in \mathbb{R},$$

where the generator $G_{\mathrm{d}} \in \mathbb{R}^{n\times n}$ is an anti-Hurwitz matrix.

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Standard dilation

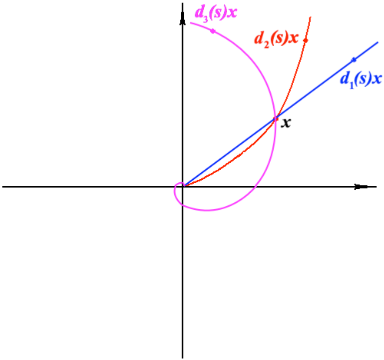
$\mathrm{d}_1(s) = e^s I, \quad G_\mathrm{d} = I \in \mathbb{R}^{n\times n}$

Weighted dilation

$\mathrm{d}_2(s) = \mathrm{diag}\{e^{r_i s}\}, G_\mathrm{d} = \mathrm{diag}\{r_i\} \succ 0$

Linear dilation

$\mathrm{d}_3(s) = e^{sG_d}$, G_d is anti-Hurwitz



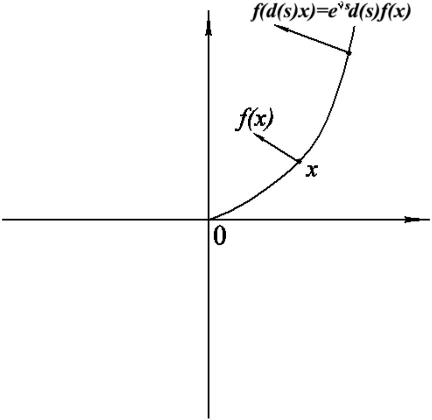
Homogeneous functions and vector fields

Definition (Homogeneous function)

A function $f : \mathbb{R}^n \to \mathbb{R}$ is d-**homogeneous** of degree ν if $f(\mathrm{d}(s)x) = e^{\nu s} f(x)$

Definition (Homogeneous vector field)

A vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ is d-**homogeneous** of degree ν if $f(\mathrm{d}(s)x) = e^{\nu s}\mathrm{d}(s) f(x)$



Geometry-preserving approximations of evolution equations

Geometric Numerical Integration (ODE/PDE→ Discrete-time):

- Finite-Difference Approximations preserving Lie Symmetries: *Dorodnitsyn 1989, Levi & Yamilov 1997, Heredero, Levi & Winternitz 2000, Bihlo & Valiquette 2017....*
- Symplectic integrators preserve some invariants of ODEs: *Channell & Scovel 1990, Leimkuhler & Reich 2004, Hairer, Wanner & Lubich 2006, ...*
- Energy preserving methods: *Quispel & McLaren 2008,...*
- Consistent discretization of ODEs (supported by ANR DIGITSLID): *Polyakov, Efimov & Brogliato 2019, Sanchez, Polyakov, Efimov 2020*

Symmetry-preserving Galerkin methods (PDE → ODE):

- Reflection-symmetry-preserving projection: *Pla et al 2015*
- Energy preserving projection: *Liu & Xing 2016*
- Dilation-symmetry-preserving projection: *Polyakov 2020*

Consistent Disctretization: Motivating Example

$$\left\{\begin{array}{l} \dot{x}(t) = u(x(t)) \\ u(x) = -2\sqrt{|x|}\mathrm{sgn}(x) \end{array}\right. \quad \overset{y=\sqrt{|x|}\mathrm{sgn}(x)}{\Leftrightarrow} \quad \left\{\begin{array}{l} \dot{y}(t) = \tilde{u}(y(t)) \\ \tilde{u}(y) \in -\mathrm{sgn}(y). \end{array}\right.$$

The explicit/implicit Euler discretization destroys the equivalence.

$$\left\{\begin{array}{l} x_{k+1} = x_k + hu_k \\ u_k = -2\sqrt{|x_{k+1}|}\mathrm{sgn}(x_{k+1}) \end{array}\right. \quad \not\Leftrightarrow \quad \left\{\begin{array}{l} y_{k+1} = y_k + h\tilde{u}_k \\ \tilde{u}_k \in -\mathrm{sgn}(y_{k+1}). \end{array}\right.$$

Consistent Disctretization: The scheme

The scheme suggested in Polyakov, Efimov, Brogliato 2019:

$\dot{x}(t)=f(x(t))$	$\overset{y=\Phi(x)}{\Leftrightarrow}$	$\dot{y}(t)=\tilde{f}(y(t))$
$\uparrow$ a consistent discrete-time approximation $\downarrow$		$\downarrow$ the implicit Euler method gives a consistent discrte-time approximation $\uparrow$
$x_{k+1}=\Phi^{-1}(\Phi(x_k)+h\tilde{f}(\Phi(x_{k+1})))$	$\overset{x_k=\Phi^{-1}(y_k)}{\Leftrightarrow}$	$y_{k+1}=y_k+h\tilde{f}(y_{k+1})$

Question: Is it possible to design a consistent explicit discretization?

II. Consistent Discretization using Lyapunov Function

Lyapunov Function of homogeneous system

$$\dot{x} = f(x), \quad t > 0, \tag{1}$$

where $f \in C(\mathbb{R}^n \backslash \{0\})$ is d-homogeneous of a degree $\nu \in \mathbb{R}$.

Theorem (Zubov 1958, Rosier 1992)

The system (1) is asymptotically stable if and only if there exist

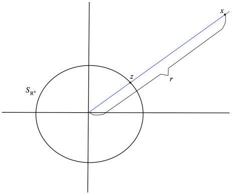
- *a* d-*homogeneous positive definite function* $V : \mathbb{R}^n \to \mathbb{R}$ *of a degree* $m > 0$, $V \in C^1(\mathbb{R}^n)$,
- *a* d-*homogeneous positive definite function* $W : \mathbb{R}^n \to \mathbb{R}$ *of the degree* $m + \nu$, $W \in C(\mathbb{R}^n)$,

such that

$$\dot{V}(x) = -W(x).$$

Polar coordinates for stable homogeneous system

$$\mathbb{R}^n \backslash \{0\} \quad \Leftrightarrow \quad \mathbb{R}_+ \times S_{\mathbb{R}^n}$$



Classical Polar Coordinates

$x = rz \in \mathbb{R}^n : \quad r = \|x\|, \quad z \in S_{\mathbb{R}^n},$
where $S_{\mathbb{R}^n}$ is the unit sphere in $\mathbb{R}^n$.

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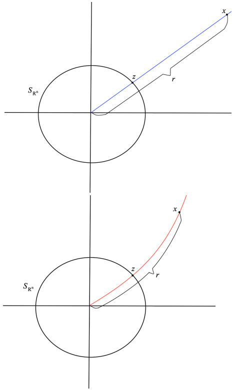
Homogeneous Polar Coordinates

$x = \mathbf{d}(\ln r) z \in \mathbb{R}^n : r = \|x\|_{\mathbf{d}}, z \in S_{\mathbb{R}^n}$
where $\|\cdot\|_{\mathbf{d}}$ - homogeneous norm.



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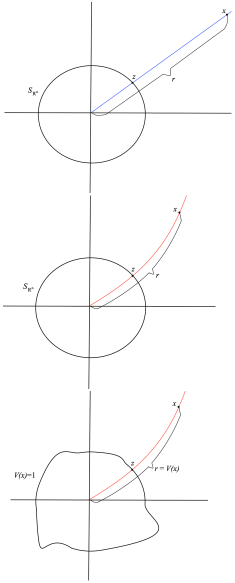
Homogeneous Polar Coordinates

$x = \mathbf{d}(\ln r) z \in \mathbb{R}^n : r = \|x\|_{\mathbf{d}}, z \in S_{\mathbb{R}^n}$
where $\|\cdot\|_{\mathbf{d}}$ - homogeneous norm.

Lyapunov Polar Coordinates

$x = \mathbf{d}(\ln V^{\frac{1}{m}}(x)) z : r = V(x), z \in S_V$
where V - homogeneous Lyapunov function of a degree $m > 0$ and S_V is the unit level set of V.



Homogenenous System in Lyapunov Polar Coordinates

$$\dot{x} = f(x), \quad t > 0, \quad f : \mathbb{R}^n \to \mathbb{R}^n \text{ is d} - \text{homogeneous}$$

Change of Variables (Lyapunov Polar Coordinates)

$$z = \mathrm{d}(-\ln V^{\frac{1}{m}}(x))x, \quad r = V(x)$$

$$\begin{cases} \dot{z} = r^{\frac{\nu}{m}} \left(f(z) - \frac{1}{m} W(z) G_{\mathrm{d}} z\right) & \text{- projected dynamics} \\ \dot{r} = -r^{\frac{m+\nu}{m}} W(z) & \text{- convergent dynamics} \end{cases} \quad (2)$$

- $z(t) \in S_V$ for all $t \geq 0$, where S_V is the unit level set of V ("sphere");
- $\inf_{z \in S_V} W(z) > 0$ and the convergence to zero is defined by the second equation $\dot{r} = -r^{\frac{k+\nu}{k}} W$, which admit the explicit solution provided that $W = W(t)$ is a known function of time.

The explicit solution of the second equation

$$\dot{r} = -r^{1+\frac{\nu}{m}} W$$

where $W = W(t)$ is assumed to be known, $m > 0, \nu \in \mathbb{R}$.

- if $\nu = 0$ then

$$r(t) = e^{-\int_{t_0}^{t} W(s)ds} r(t_0), \quad t \geq t_0.$$

- if $\nu > 0$ then

$$r(t) = \frac{r(t_0)}{\left(1 + \frac{\nu}{m} r^{\frac{\nu}{m}}(t_0) \int_0^t W(s)ds\right)^{\frac{m}{\nu}}}, \quad t \geq t_0,$$

- if $\nu < 0$ then

$$r(t) = \begin{cases} \left(r^{\frac{-\nu}{m}}(t_0) + \frac{\nu}{m} \int_0^t W(s)ds\right)^{\frac{m}{-\nu}} & \text{if } r^{\frac{-\nu}{m}}(t_0) > \frac{-\nu}{m} \int_0^t W(s)ds, \\ 0 & \text{if } r^{\frac{-\nu}{m}}(t_0) \leq \frac{-\nu}{m} \int_0^t W(s)ds \end{cases}$$

Consistent Discretization using explicit Euler method

Explicit discretization of the first equation in

$$\begin{cases} \dot{z} = r^{\frac{\nu}{m}} \left(f(z) - \frac{1}{m} W(z) G_{\mathrm{d}} z \right), \\ \dot{r} = -r^{\frac{m+\nu}{m}} W(z), \end{cases}$$

gives

$$\begin{cases} \frac{z_{k+1} - z_k}{h} = r_k^{\frac{\nu}{m}} \left(f(z_k) - \frac{1}{m} W(z_k) G_{\mathrm{d}} z_k \right), \\ \dot{r} = -r^{\frac{m+\nu}{m}} W(z_k), t \in [t_k, t_{k+1}], \end{cases}$$

where $r_k \approx r(t_k)$, $z_k \approx z(t_k)$.

Consistent Discretization using explicit Euler method

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where $r_k \approx r(t_k)$, $z_k \approx z(t_k)$. Since $z(t) \in S_V, \forall t \geq 0$ then

$$z_{k+1} = P\left(z_k + h r_k^{\frac{\nu}{m}} \left(f(z_k) - \frac{1}{m} W(z_k) G_{\mathrm{d}} z_k \right) \right) \qquad (3)$$

where $P(z) = \mathrm{d}(-\ln V^{\frac{1}{k}}(z))z$ - homogeneous projector on S_V.

Consistent Discretization using explicit Euler method

Explicit discretization of the first equation in

$$\begin{cases} \dot{z} = r^{\frac{\nu}{m}} \left(f(z) - \frac{1}{m} W(z) G_{\mathrm{d}} z \right), \\ \dot{r} = -r^{\frac{m+\nu}{m}} W(z), \end{cases}$$

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$$\begin{cases} \frac{z_{k+1}-z_k}{h} = r_k^{\frac{\nu}{m}} \left(f(z_k) - \frac{1}{m} W(z_k) G_{\mathrm{d}} z_k \right), \\ \dot{r} = -r^{\frac{m+\nu}{m}} W(z_k), t \in [t_k, t_{k+1}], \end{cases}$$

where $r_k \approx r(t_k)$, $z_k \approx z(t_k)$. Since $z(t) \in S_V, \forall t \geq 0$ then

$$z_{k+1} = P\left(z_k + h r_k^{\frac{\nu}{m}} \left(f(z_k) - \frac{1}{m} W(z_k) G_{\mathrm{d}} z_k \right) \right) \quad (3)$$

where $P(z) = \mathrm{d}(-\ln V^{\frac{1}{k}}(z))z$ - homogeneous projector on S_V. The exact discretization of the second equation for $\nu < 0$ is given by

$$r_{k+1} = \begin{cases} \left(r_k^{\frac{-\nu}{m}} + \frac{\nu}{m}(t_{k+1} - t_k) W(z_k) ds \right)^{\frac{m}{-\nu}} & \text{if} \quad r_k^{\frac{-\nu}{m}} > \frac{-\nu}{m}(t_{k+1}-t_k)W(z_k), \\ 0 & \text{if } r^{\frac{-\nu}{m}}(t_0) \leq \frac{-\nu}{m}(t_{k+1}-t_k)W(z_k). \end{cases}$$

For $\nu \geq 0$ the system can be discretized similarly.

III. Examples

Consistent discretization of 2-SM controller

Consider quasi-continuous 2-SM control system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = u, \quad u = -k_1 \frac{x_1 + k_2 x_2 |x_2|}{|x_1| + k_2 |x_2|^2}, \quad k_1, k_2 > 0$$

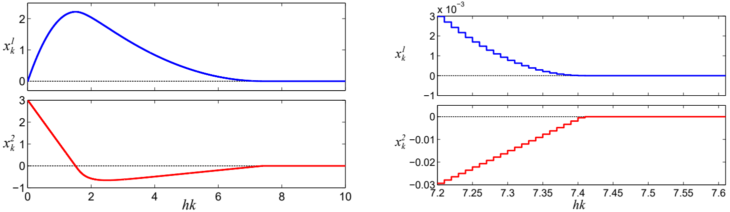
which is d-homogeneous of the degree $\nu = -1$ for $\mathrm{d}(s) = \mathrm{diag}\{e^{2s}, e^{s}\}$. For any $k_1 > 0$ there exists $k_2, \alpha > 0$ such that the system has a d-homogeneous Lyapunov function V of the degree $m = 3$ given by [1]

$$V(x) = \alpha \frac{2}{3} |x_1|^{\frac{3}{2}} + x_1 x_2 + \frac{1}{3} k_2 |x_2|^3$$

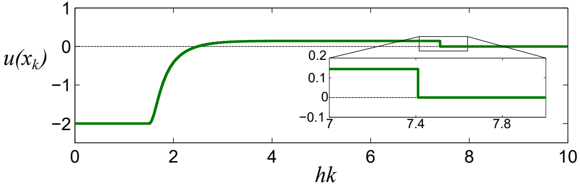
such that $\dot{V}(x) = -W(x)$ with

$$W(x) = k_1 \frac{(x_1 + k_2 x_2 |x_2|)^2}{|x_1| + k_2 |x_2|^2} - \alpha \sqrt{|x_1|}\, \mathrm{sign}(x_1) x_2 - x_2^2$$

[1] Sanchez & Moreno 2019



States



Control input

Consistent discretization of a positive degree system

Consider the following system

$$\dot{x}_1 = -k_1 x_1 \sqrt{|x_1|} + x_2, \qquad \dot{x}_2 = -k_2 x_2 |x_2|, \quad k_1, k_2 > 0$$

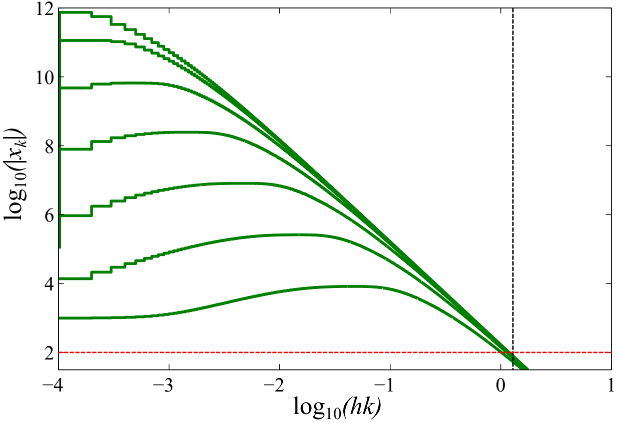
which is d-homogeneous of the degree $\nu = 1$ for $\mathrm{d}(s) = \mathrm{diag}\{e^s, e^{2s}\}$. For any $k_1 > 0$ there exists $k_2, \alpha > 0$ such that the system has a d-homogeneous Lyapunov function V of the degree $m = 5$ given by

$$V(x) = \alpha \frac{2}{5} |x_1|^{\frac{5}{2}} - x_1 x_2 + \frac{3}{5} k_2 |x_2|^{\frac{5}{3}}$$

such that $\dot{V}(x) = -W(x)$ with

$$W(x) = \left(k_1 x_1 \sqrt{|x_1|} - x_2\right)^2 + k_2 \left(\alpha x_1 |x_1| |x_2|^{\frac{2}{3}} \operatorname{sign}(x_2) - |x_1|^3\right)$$

Simulation results for the system with positive degree



States

Summary

- A new method for consistent discretization of homogeneous systems is developed.
- It allows **explicit** consistent discretization schemes to be designed.
- Theoretical results are supported by numerical simulations.

Thank you very much for your attension