Non-Holonomic constraints of a wheel

W. Perruquetti
Master AG2I

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Abstract

The aim is to give the non holonomic constraints of a wheel (pure rolling: no slipping, no skidding).

1 Preliminaries

Question 1: Let us consider two frames in the plane: $\mathcal{F}_i = (O_i, \overrightarrow{x}_i, \overrightarrow{y}_i)$ and $\mathcal{F}_j = (O_j, \overrightarrow{x}_j, \overrightarrow{y}_j)$ (the $\overrightarrow{z}$ axis being the same for both frame $\odot$). Let us denote by θ_{ij} the rotation angle $(\widehat{\overrightarrow{x}_i, \overrightarrow{x}_j})$. Let M be a point in the plane, in stead of writing $m_{/\mathcal{F}_i}$ we write $m_{/i}$ to denotes its coordinates with respect to the frame $\mathcal{F}_i$. Denoting the homogeneous coordinates of the point M in the plane by

$$M = \begin{pmatrix} m_x \\ m_y \\ 1 \end{pmatrix}$$

prove that

$$M_{/i} = T_i^j M_{/j}$$

where the homogeneous transformations in the plane is given by

$$T_i^j = \begin{pmatrix} R_i^j & P_i^j \\ (0)_{(2\times 2)} & 1 \end{pmatrix}$$

where P_i^j is the coordinates of O_j w.r.t. $\mathcal{F}_i$:

$$P_i^j = \begin{pmatrix} o_{j,x} \\ o_{j,y} \end{pmatrix}_{/i}$$

and

$$R_i^j = \begin{pmatrix} \mathrm{c}_{ij} & -\mathrm{s}_{ij} \\ \mathrm{s}_{ij} & \mathrm{c}_{ij} \end{pmatrix},$$

where the following standard notations are used $\mathrm{c}_{ij} = \cos(\theta_{ij}), \mathrm{s}_{ij} = \sin(\theta_{ij})$.

Answer 1: We have

$$\overrightarrow{O_jM}_{/j} = m_{x/j}\overrightarrow{x}_j + m_{y/j}\overrightarrow{y}_j$$

By definition of the angle $\theta_{ij} = (\widehat{\overrightarrow{x}_i, \overrightarrow{x}_j})$ we have

$$\begin{array}{rcl} \overrightarrow{x}_j & = & \cos(\theta_{ij})\overrightarrow{x}_i + \sin(\theta_{ij})\overrightarrow{y}_i \\ \overrightarrow{y}_j & = & -\sin(\theta_{ij})\overrightarrow{x}_i + \cos(\theta_{ij})\overrightarrow{y}_i \end{array}$$

Since $\overrightarrow{O_iM} = \overrightarrow{O_iO_j} + \overrightarrow{O_jM}$, thus

$$\begin{pmatrix} m_{x/i} \\ m_{y/i} \end{pmatrix} = \begin{pmatrix} o_{j,x} \\ o_{j,y} \end{pmatrix}_{/i} + R_i^j \begin{pmatrix} m_{x/j} \\ m_{y/j} \end{pmatrix}$$

leading to

$$M_{/i} = T_i^j M_{/j}$$

Let us consider a mobile robot and its localization with respect to a fixed frame $\mathcal{F} = (O, \overrightarrow{x}, \overrightarrow{y})$: a mobile frame is attached to the robot $\mathcal{F}_m = (O_m, \overrightarrow{x}_m, \overrightarrow{y}_m)$ (the $\overrightarrow{z}$ axis being the same for both frame $\odot$):

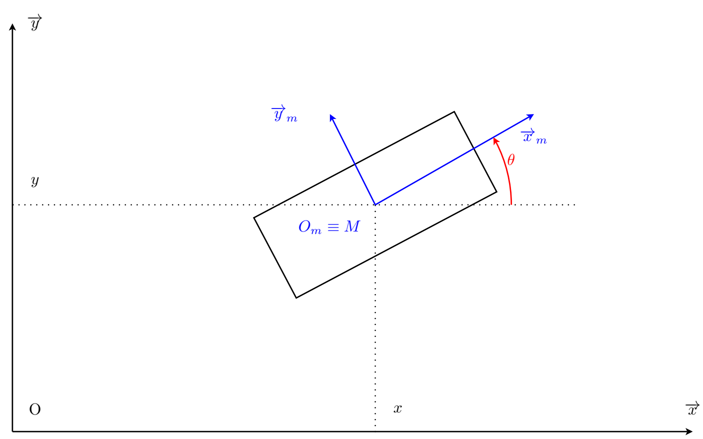

Figure 1: Mobile Robot

When there is no confusion, the z coordinate is omitted if the considered point is in the plane. The mobile robot posture is defined by:

$$P = \begin{pmatrix} x \\ y \\ \theta \end{pmatrix}_{/\mathcal{F}}$$

Question 1: Prove that

$$\dot{P}_{/\mathcal{F}_m} = \begin{pmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \dot{P}_{/\mathcal{F}}$$

Answer 2: Direct proof (without using question 1). Using

$$\begin{aligned} P &= x\overrightarrow{x} + y\overrightarrow{y} + \theta\overrightarrow{z} \\ &= \begin{pmatrix} x \\ y \\ \theta \end{pmatrix}_{/\mathcal{F}} \end{aligned}$$

On one hand (since the frame $\mathcal{F}$ is fixed)

$$\dot{P}_{/\mathcal{F}} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix}_{/\mathcal{F}}$$

on the ohter hand

$$\dot{P}_{/\mathcal{F}_m} = \begin{pmatrix} \dot{x}_{/m} \\ \dot{y}_{/m} \\ \dot{\theta} \end{pmatrix}_{/\mathcal{F}_m} = \dot{x}_{/m}\overrightarrow{x}_m + \dot{y}_{/m}\overrightarrow{y}_m + \dot{\theta}\overrightarrow{z}$$

and

$$\begin{aligned} \overrightarrow{x}_m &= \cos(\theta)\overrightarrow{x} + \sin(\theta)\overrightarrow{y} \\ \overrightarrow{y}_m &= -\sin(\theta)\overrightarrow{x} + \cos(\theta)\overrightarrow{y} \end{aligned}$$

thus

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}_{/\mathcal{F}} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} \dot{x}_{/m} \\ \dot{y}_{/m} \end{pmatrix}_{/\mathcal{F}_m}$$

Leading to the desired result. An other proof is based on question 1: since

$$\begin{aligned} m_{/\mathcal{F}} &= R_{\mathcal{F}}^{\mathcal{F}_m} m_{/\mathcal{F}_m}, \\ R_{\mathcal{F}}^{\mathcal{F}_m} &= R(\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \end{aligned}$$

we have $m_{/\mathcal{F}_m} = R^T(\theta) m_{/\mathcal{F}}$. Using the homogeneous transformation $T(\theta)$ associated to matrix $R^T(\theta)$:

$$\begin{aligned} \begin{pmatrix} \dot{x}_{/m} \\ \dot{y}_{/m} \\ \dot{\theta} \end{pmatrix}_{/\mathcal{F}_m} &= T(\theta) \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix}_{/\mathcal{F}} \\ T(\theta) &= \begin{pmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

2 Fixed Centered Wheel

Let us consider a fixed centered wheel (see figure 2)

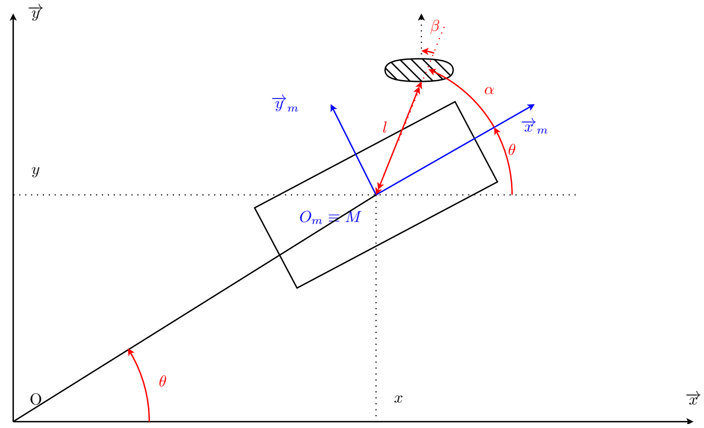

Figure 2: Fixed Centered Wheel

The wheel/ground contact point is denoted by A. The position of A in the mobile frame $(O_m, \vec{x}_m, \vec{y}_m)$ attached to the robot, is characterized using polar coordinates by the distance $O_mA = l$ and the angle α. The orientation of the plane of the wheel with respect to O_mA is characterized by the constant angle β. The time varying rotation angle of the wheel around its (horizontal) axle is denoted $\phi(t)$, the wheel rotation axis is given by the dashed vector in the figure $(\vec{y_r})$ and the radius of the wheel is denoted r. The position of the wheel is thus characterized by 4 constants (α, β, l, r) and its motion by a time varying angle $\phi(t)$.

Let us denote the moving frame attached to the wheel by $\mathcal{F}_w = (O_w \equiv A, \vec{x}_w, \vec{y}_w)$ (the $\vec{z}$ axis being the same for all frames $\odot$).

Question 2: Let us denote by P the wheel center. Give the coordinates of the points A, P within the frames $\mathcal{F}_w, \mathcal{F}_m$ and finally $\mathcal{F}$.

Answer 3: Let us find T^m and T^w_m :

$$T^m = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & x \\ \sin(\theta) & \cos(\theta) & y \\ 0 & 0 & 1 \end{pmatrix}$$

$$T^w_m = \begin{pmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) & l\cos(\alpha) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) & l\sin(\alpha) \\ 0 & 0 & 1 \end{pmatrix}$$

Thus, for the point A :
$$\begin{pmatrix} a_x \\ a_y \\ 1 \end{pmatrix}_{/\mathcal{F}_w} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{aligned}
\begin{pmatrix} a_x \\ a_y \\ 1 \end{pmatrix}_{/\mathcal{F}_m} &= T_m^w \begin{pmatrix} a_x \\ a_y \\ 1 \end{pmatrix}_{/\mathcal{F}_w} \\
&= \begin{pmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) & l\cos(\alpha) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) & l\sin(\alpha) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} l\cos\alpha \\ l\sin\alpha \\ 1 \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
\begin{pmatrix} a_x \\ a_y \\ 1 \end{pmatrix}_{/\mathcal{F}} &= T^m \begin{pmatrix} a_x \\ a_y \\ 1 \end{pmatrix}_{/\mathcal{F}_m} \\
&= \begin{pmatrix} \cos(\theta) & -\sin(\theta) & x \\ \sin(\theta) & \cos(\theta) & y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} l\cos\alpha \\ l\sin\alpha \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} x + l\cos(\alpha+\theta) \\ y + l\sin(\alpha+\theta) \\ 1 \end{pmatrix}
\end{aligned}$$

Since the $\overrightarrow{z}$ vector is common to all frame, the coordinates of the point P are:$(0,0,r)^T_{/\mathcal{F}_w}$, $(l\cos\alpha, l\sin\alpha, r)^T_{/\mathcal{F}_m}$, $(x + l\cos(\alpha+\theta), y + l\sin(\alpha+\theta), r)^T_{/\mathcal{F}}$.

Question 3: Assuming that the wheel is rolling without slipping and skidding (pure rolling), find the non-holonomic constraints in the frame $\mathcal{F}, \mathcal{F}_m$ and finally $\mathcal{F}_w$.

Answer 4: For a pure rolling wheel, the relative speed at the wheel/ground contact point A is null: $\overrightarrow{v}_A = 0$. This leads to a relation between $\overrightarrow{v}_P$ the speed of the centre of the wheel P and $\overrightarrow{\omega}$ the rotational speed vector of the wheel with respect to the ground:

$$\overrightarrow{v}_A = \overrightarrow{0} = \overrightarrow{v}_P + \overrightarrow{\omega} \wedge \overrightarrow{PA}.$$

First let us note that what

$$\overrightarrow{PA}_{/\mathcal{F}} = \overrightarrow{PA}_{/\mathcal{F}_m} = \overrightarrow{PA}_{/\mathcal{F}_w} = \begin{pmatrix} 0 \\ 0 \\ -r \end{pmatrix}.$$

In the frame $\mathcal{F}$, from question 2 we have $\overrightarrow{OP}_{/\mathcal{F}} = (x + l\cos(\alpha+\theta), y + l\sin(\alpha + \theta), r)^T_{/\mathcal{F}}$ thus

$$\begin{aligned}
\overrightarrow{v}_{P/\mathcal{F}} &= (\dot{x} - l\sin(\alpha+\theta)\dot{\theta})\overrightarrow{x} + (\dot{y} + l\cos(\alpha+\theta)\dot{\theta})\overrightarrow{y} \\
\overrightarrow{v}_{P/\mathcal{F}_m} &= \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \overrightarrow{v}_{P/\mathcal{F}} \\
&= \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} \dot{x} - l\sin(\alpha+\theta)\dot{\theta} \\ \dot{y} + l\cos(\alpha+\theta)\dot{\theta} \end{pmatrix} \\
&= \begin{pmatrix} (\cos(\theta)\dot{x} + \sin(\theta)\dot{y}) - l\dot{\theta}\sin\alpha \\ (-\sin(\theta)\dot{x} + \cos(\theta)\dot{y}) + l\dot{\theta}\cos\alpha \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & -l\sin\alpha \\ 0 & 1 & l\cos\alpha \end{pmatrix} \begin{pmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix} \\
\overrightarrow{v}_{P/\mathcal{F}_w} &= \begin{pmatrix} \cos(\alpha+\beta) & \sin(\alpha+\beta) \\ -\sin(\alpha+\beta) & \cos(\alpha+\beta) \end{pmatrix} \begin{pmatrix} 1 & 0 & -l\sin\alpha \\ 0 & 1 & l\cos\alpha \end{pmatrix} \overrightarrow{v}_{P/\mathcal{F}_m} \\
&= \begin{pmatrix} \cos(\alpha+\beta) & \sin(\alpha+\beta) & l\sin\beta \\ -\sin(\alpha+\beta) & \cos(\alpha+\beta) & l\cos\beta \end{pmatrix} \begin{pmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix}
\end{aligned}$$

The rotational vector has two parts one comming from the θ rotation around the $\overrightarrow{z}$ axis and the other one comming from the wheel ϕ rotation around the axis $\overrightarrow{y}_w$:

$$\overrightarrow{\omega} =$$