Robotics: Modeling (Serial)

(option ISD-master SMART)

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Objectives

Pre-requisits:

- Basic physics and maths,
- Chapter 1: Robotics: Introduction.
- Chapter 2: Modeling (WMR).

Objectives:

- Rigid-Body Mechanics: frame, points, rigid-body transformation, rotations, change of coordinates, euler angles, . . .
- Robot description: DH, geometric model
- Modeling : kinematic and dynamic models.

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Introduction

Manipulators are divided into two classes:

- serial robot (open kinematic chain),
- parallel robot (closed kinematic chain),

Here only serial robots that is robots having open kinematic chains (every links in the chain is used one and only one times!)

Rigid-Body Mechanics

- rigid = "indeformable";
- If A and B are two different points from the body then $\frac{d\overrightarrow{AB}}{dt} = 0$ or alternatively $\overrightarrow{AB} = cte$
- Volume is constant.
- liaisons articulées par des joints mécaniques (parfaits: pas de jeux ni de flexibilité)

Rigid-Body Mechanics

- Repérage d'un corps rigide libre dans l'espace
  - spécifier la position + l'orientation
  - au moyen de la position de 3 points non colinéaires
  - 3 relations de liaisons (distance fixe entre les points)
- 6 degrés de liberté (DDL):
  - 3 ddl de translation : position d'un point de référence dans un système de coordonnées
  - 3 ddl de rotation : paramètres de rotation: angles d'Euler, de Bryant, etc.

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Rigid-Body Mechanics: Mathematical Background

notations

$\mathcal{R} = (O; x; y; z)$ is a Cartesian right-hand orthonormal frame, with Gibbs convention.

With respect to a fixed origin O, the position of a point P is described by $\overrightarrow{OP}$ or simply p

- Point P position : vector p with coordinates in $\mathbb{R}^3$:

$$p = \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix}$$

- Point Motion $p(t)$: parametric curve in $\mathbb{R}^3$.
- Point Path $p(s)$: geometric path associated to the motion ($s \in [01]$ normalized curvilinear abscissa).

Rigid-Body Mechanics: Mathematical Background

Notation

Sometimes the coordinates of a point have to de specified w.r.t. different frames in that case (when needed) when denote a vector p in frame $\mathcal{F}$ by $p_{/\mathcal{F}}$. And its coordinates are denoted by

$$p_{/\mathcal{F}} = \begin{pmatrix} p_{x/\mathcal{F}} \\ p_{y/\mathcal{F}} \\ p_{z/\mathcal{F}} \end{pmatrix}.$$

A frame is "repère" in french thus sometimes frames are also denoted by $\mathcal{R}, \mathcal{R}_i, \mathcal{R}' \ldots$. When a frame is denoted by $\mathcal{F}_i$ or $\mathcal{R}_i$ in stead of writing $p_{/\mathcal{F}_i}$ we write $p_{/i}$. For example $p_{/0}$ means the coordinates of the point P w.r.t. the frame $\mathcal{F}_0$.

Rigid-Body Mechanics: Mathematical Background

- Rigid body : for any pair of points with coordinates m and n :

$$\|m(t) - n(t)\| = \|m(0) - n(0)\| = constant$$

- Rigid-body pose : position and orientation of a frame attached to this rigid body in $\mathcal{R}$
- Rigid-body transformation : result of a rigid motion.
- rigid-body transformation = Application which preserves distances and orientations.

Consequence

In a rigid-body transformation, a right-hand orthonormal frame is changed into another right-hand orthonormal frame.

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Rigid-Body Mechanics: Mathematical Background

- $\mathcal{F} = (O; x; y; z)$ and a new frame (same origin) $\mathcal{F}' = (O; x'; y'; z')$: right-hand orthonormal frame
- $x'_{/\mathcal{F}}; y'_{/\mathcal{F}}; z'_{/\mathcal{F}}$ denotes the coordinates of the vectors $x'; y'; z'$ in the frame $\mathcal{F}$

$$x'_{/\mathcal{F}} = \begin{pmatrix} x' \cdot x \\ x' \cdot y \\ x' \cdot z \end{pmatrix}; y'_{/\mathcal{F}} = \begin{pmatrix} y' \cdot x \\ y' \cdot y \\ y' \cdot z \end{pmatrix}; z'_{/\mathcal{F}} = \begin{pmatrix} z' \cdot x \\ z' \cdot y \\ z' \cdot z \end{pmatrix}$$

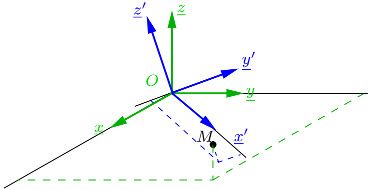
Definition

$R_{\mathcal{F}}^{\mathcal{F}'} = (x'_{/\mathcal{F}} y'_{/\mathcal{F}} z'_{/\mathcal{F}})$ of dimension 3×3 is the rotation matrix from frame $\mathcal{F}$ to frame $\mathcal{F}'$ (change of basis). It is a matrix whose columns are the vectors of the final frame expressed in the initial frame.

Rigid-Body Mechanics: Mathematical Background

Role of the rotation matrix

- gives a representation of the rotation of a frame attached to a rigid body, from frame $\mathcal{F}$ to frame $\mathcal{F}'$
- allows to calculate the coordinates of a point in a new frame



Rotation matrix

Rigid-Body Mechanics: Mathematical Background

Let $\mathcal{R}_0 = (O; x_0; y_0; z_0)$ and $\mathcal{R}_1 = (O; x_1; y_1; z_1)$ be two frames then

$$R_0^1 = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix}$$

Let $m_{/0} = (m_{x/0} m_{y/0} m_{z/0})_{/0}^T$ and $m_{/1} = (m_{x/1} m_{y/1} m_{z/1})_{/1}^T$ be the coordinates of point M respectively in $\mathcal{R}_0$ and $\mathcal{R}_1$.

Rigid-Body Mechanics: Mathematical Background

Then :

$$m_{/0} = m_{x/1}x_{1/0} + m_{y/1}y_{1/0} + m_{z/1}z_{1/0} = \begin{pmatrix} x_{1/0} & y_{1/0} & z_{1/0} \end{pmatrix} \begin{pmatrix} m_{x/1} \\ m_{y/1} \\ m_{z/1} \end{pmatrix}$$

Consequence

Change of basis equation (or coordinate transformation) :

$$m_{/0} = R_0^1 m_{/1}.$$

Rigid-Body Mechanics: Mathematical Background

Clearly from $m_{/0} = R_0^1 m_{/1}$, one deduce

$$x_{1/0} = R_0^1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, y_{1/0} = R_0^1 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} z_{1/0} = R_0^1 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

leading to

$$R_0^1 = (x_{1/0} y_{1/0} z_{1/0})$$

$$R_0^1 = \begin{pmatrix} x_1 \cdot x_0 & y_1 \cdot x_0 & y_1 \cdot x_0 \\ x_1 \cdot y_0 & y_1 \cdot y_0 & y_1 \cdot y_0 \\ x_1 \cdot z_0 & y_1 \cdot z_0 & y_1 \cdot z_0 \end{pmatrix}$$

Rigid-Body Mechanics: Mathematical Background

We have

$$R_0^1 = (x_{1/0} y_{1/0} z_{1/0}) = \begin{pmatrix} x_{0/1} \\ y_{0/1} \\ z_{0/1} \end{pmatrix} = R_1^{0T}$$

$$R_0^1 = \left(R_1^0\right)^T$$

Inverse of Rotation Matrix (orthonormal matrix)

$$\left(R_0^1\right)^{-1} = R_0^{1T}$$

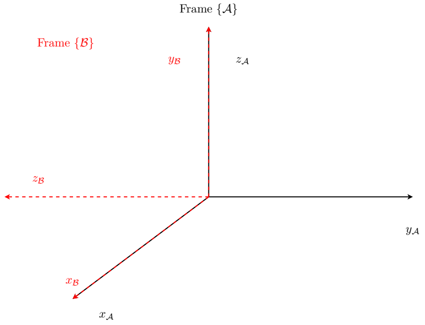
Rigid-Body Mechanics: Mathematical Background

Identity matrix is denoted as I, whatever the dimension.

- Orthogonality : $R^T R = I$ and $\det R = 1$.
- Neutral element : identity matrix of dimension 3.
- Unique inverse : $R^{-1} = R^T$.
- Combination of two successive rotations R_1 and R_2 : rotation $R_1 R_2$.
- $SO(3)$ non commutative group (multiplication)

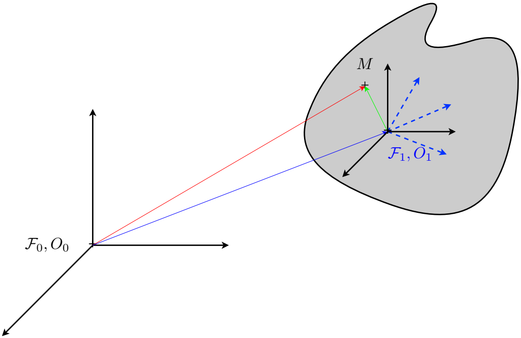
Rigid-Body Mechanics: Mathematical Background

Example:



$$R_{\mathcal{A}}^{\mathcal{B}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

Rigid-Body Mechanics: Mathematical Background

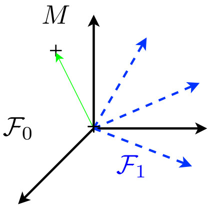


Frame : $\mathcal{F}_1$: ${\color{blue}x_{1/0}, y_{1/0}, z_{1/0}}, M_{\text{solid}/0}$

$$\{\mathcal{F}_1\} = \{R_0^1 \; M_{\text{solid}/0}\}$$

Rigid-Body Mechanics: Mathematical Background

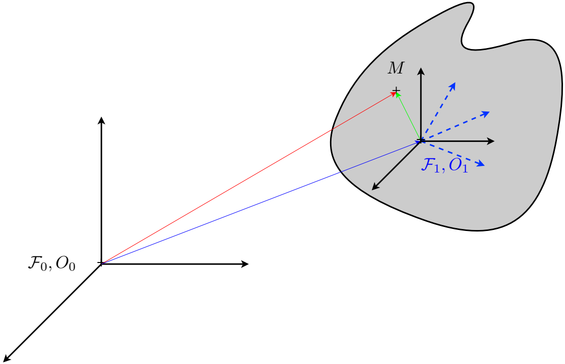
From one frame to the other one we have : a translation and a rotation.
Rotations: the two frames have the same origin $O_0 = O_1$.



If a point M is given in frame $\{\mathcal{F}_1\}$

$$m_{/0} = R_0^1 m_{/1}$$

Rigid-Body Mechanics: Mathematical Background



The position of the point M is described in the two frames.

$$\overrightarrow{O_0M} \Rightarrow \overrightarrow{O_1M}$$

Rigid-Body Mechanics: Mathematical Background

Homogeneous transformation

We have

$$\overrightarrow{O_0M} = \overrightarrow{O_0O_1} + \overrightarrow{O_1M},$$

let us note that for a robot $\overrightarrow{O_0O_1} = P_{O_1/0}$ denotes the position of the organ w.r.t frame $\{\mathcal{F}_0\}$. Since $\overrightarrow{O_1M}_{/0} = R_0^1\overrightarrow{O_1M}_{/1}$, it leads to

$$m_{\text{solid}/0} = R_0^1 m_{\text{solid}/1} + P_{O_1/0}$$

$$\begin{pmatrix} m_{\text{solid}/0} \\ 1 \end{pmatrix} = \begin{pmatrix} R_0^1 & P_{O_1/0} \\ (0)_{1\times 3} & 1 \end{pmatrix} \begin{pmatrix} m_{\text{solid}/1} \\ 1 \end{pmatrix} \tag{1}$$

Rigid-Body Mechanics: Mathematical Background

Homogeneous transformation

$$M_{/0} = T_0^1 M_{/1}$$

Homogeneous vector

$$M_{/0} = \begin{pmatrix} m_{\text{solid}/1} \\ 1 \end{pmatrix} \in \mathbb{R}_4.$$

Homogeneous transformation

$$T_0^1 = \begin{pmatrix} R_0^1 & P_{O_1/0} \\ (0)_{1\times 3} & 1 \end{pmatrix} \in \mathbb{R}_{4\times 4}.$$

Rigid-Body Mechanics: Mathematical Background

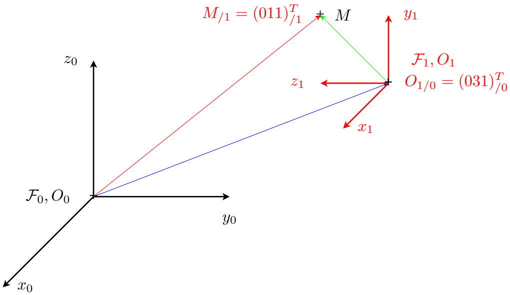
Be careful $R^{-1} = R^T$ but $T^{-1} \neq T^T$.
Inverse Homogeneous transform

$$\left(T_0^1\right)^{-1} = \begin{pmatrix} \left(R_0^1\right)^T & -\left(R_0^1\right)^T P_{O_1/0} \\ (0)_{1\times 3} & 1 \end{pmatrix}.$$

Note thate $P_{O_0/1} = -\left(R_0^1\right)^T P_{O_1/0}$.

Rigid-Body Mechanics: Mathematical Background

Example



$$T_0^1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$M_{\text{solid}/1} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

Since $M_{/0} = T_0^1 M_{/1}$, we get

$$M_{\text{solid}/0} = \begin{pmatrix} 0 \\ 2 \\ \end{pmatrix}$$

Rigid-Body Mechanics: Mathematical Background

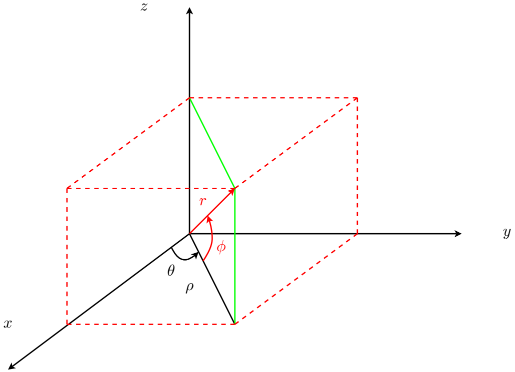
Homogeneous transformation composition We clearly have

$$T_0^4 = T_0^1 T_1^2 T_2^4 T_3^4$$

Transformation to get object from frame 4 into the frame 0 !!

Rigid-Body Mechanics: Mathematical Background

Position Representation



- Cartesian : (x, y, z)
- Cylindrical : (x, y, z)
- Spherical : (x, y, z)

Rigid-Body Mechanics: Mathematical Background

Rotation Representation

Rotation Matrix

$$R = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix} = (r_1, r_2, r_3)$$

Direction cosines

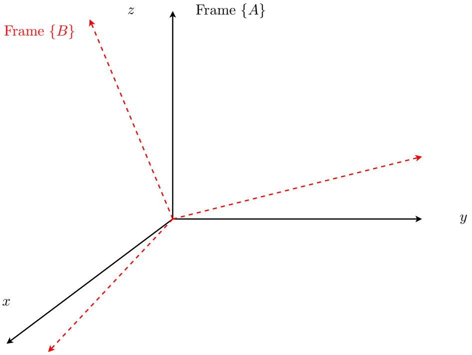
$$x_r = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix}_{(9\times 1)}$$

Constraints: $\|r_1\| = \|r_2\| = \|r_3\| = 1$

$$r_i \cdot r_j = \delta_{ij}$$

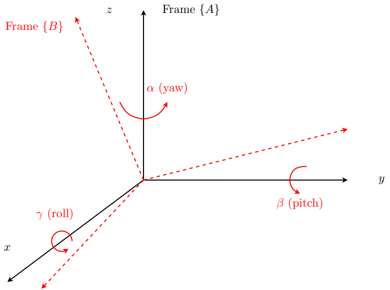
Rigid-Body Mechanics: Mathematical Background

Three Angle Representations



One can go from one frame $\{\mathcal{A}\}$ to the other one $\{\mathcal{B}\}$ by different elementary rotations around axes. For example $(Z - Y - X)$ means first rotation around Z, then around Y and lastly around X.

Rigid-Body Mechanics: Mathematical Background



$$R_{\mathcal{B}}^{\mathcal{A}} = R_Z(\alpha) R_Y(\beta) R_X(\gamma)$$

$$R_{\mathcal{B}}^{\mathcal{A}} = \begin{pmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & c\gamma & -s\gamma \\ 0 & s\gamma & c\gamma \end{pmatrix}$$

Rigid-Body Mechanics: Mathematical Background

Inverse problem

$$R_{\mathcal{B}}^{\mathcal{A}} = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix} \tag{2}$$

$$= \begin{pmatrix} \mathrm{c}\alpha \cdot \mathrm{c}\beta & \mathrm{c}\alpha \cdot \mathrm{s}\beta \cdot \mathrm{s}\gamma - \mathrm{s}\alpha \cdot \mathrm{c}\gamma & \mathrm{c}\alpha \cdot \mathrm{s}\beta \cdot \mathrm{c}\gamma + \mathrm{s}\alpha \cdot \mathrm{s}\gamma \\ \mathrm{s}\alpha \cdot \mathrm{c}\beta & \mathrm{s}\alpha \cdot \mathrm{s}\beta \cdot \mathrm{s}\gamma + \mathrm{c}\alpha \cdot \mathrm{c}\gamma & \mathrm{s}\alpha \cdot \mathrm{s}\beta \cdot \mathrm{c}\gamma - \mathrm{c}\alpha \cdot \mathrm{s}\gamma \\ -\mathrm{s}\beta & \mathrm{c}\beta \cdot \mathrm{s}\gamma & \mathrm{c}\beta \cdot \mathrm{c}\gamma \end{pmatrix} \tag{3}$$

$$\left.\begin{array}{c} \cos\beta = \mathrm{c}\beta = \sqrt{r_{11}^2 + r_{21}^2} \\ \sin\beta = \mathrm{s}\beta = -r_{31} \end{array}\right\} \Rightarrow \beta = \arctan 2\left(-r_{31}, \sqrt{r_{11}^2 + r_{21}^2}\right) \tag{4}$$

if $\mathrm{c}\beta = 0$ this is $\beta \equiv \frac{\pi}{2}(\pi) \Rightarrow$ singularity of the representation

Rigid-Body Mechanics: Mathematical Background

Only $(\alpha+\gamma)$ or $(\alpha-\gamma)$ is defined

$\underline{\mathrm{c}\beta=0,\mathrm{s}\beta=1:}$

$$R_{\mathcal{B}}^{\mathcal{A}}=\begin{pmatrix} 0 & -\mathrm{s}(\alpha-\gamma) & \mathrm{c}(\alpha-\gamma) \\ 0 & \mathrm{c}(\alpha-\gamma) & \mathrm{s}(\alpha-\gamma) \\ -1 & 0 & 0 \end{pmatrix}$$

$\underline{\mathrm{c}\beta=0,\mathrm{s}\beta=-1:}$

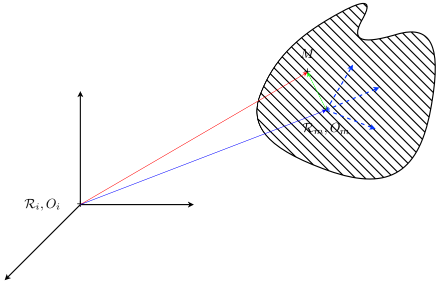
$$R_{\mathcal{B}}^{\mathcal{A}}=\begin{pmatrix} 0 & -\mathrm{s}(\alpha+\gamma) & -\mathrm{c}(\alpha+\gamma) \\ 0 & \mathrm{c}(\alpha+\gamma) & -\mathrm{s}(\alpha+\gamma) \\ 1 & 0 & 0 \end{pmatrix}$$

Rigid-Body Mechanics: Mathematical Background

Change of frame

Solide dans l'espace (par example satellite) : Changement de repère sur la dérivation temporelle d'un vecteur

$$\left(\frac{d\overrightarrow{x}}{dt}\right)_{\mathcal{R}_i} = \left(\frac{d\overrightarrow{x}}{dt}\right)_{\mathcal{R}_m} + \overrightarrow{\omega}_{\mathcal{R}_m/\mathcal{R}_i} \wedge \overrightarrow{x},$$



Rigid-Body Mechanics: Mathematical Background

Change of frame

Ainsi, si M est un point lié à un solide en rotation auquel est attaché $\mathcal{R}_m$ un repère en mouvement d'origine O_m (cf. figure) :

$$\begin{aligned}\left(\frac{d\overrightarrow{O_iM}}{dt}\right)_{\mathcal{R}_i} &= \left(\frac{d\overrightarrow{O_iO_m}}{dt}\right)_{\mathcal{R}_i} + \left(\frac{d\overrightarrow{O_mM}}{dt}\right)_{\mathcal{R}_i} \\ &= \left(\frac{d\overrightarrow{O_iO_m}}{dt}\right)_{\mathcal{R}_i} + \left(\frac{d\overrightarrow{O_mM}}{dt}\right)_{\mathcal{R}_m} + \overrightarrow{\omega}_{\mathcal{R}_m/\mathcal{R}_i} \wedge \overrightarrow{O_mM},\end{aligned}$$

avec $\mathcal{R}_i$ un repère inertiel d'origine O_i (pour un corps rigide M et O_m est une distance constante !)

$$\overrightarrow{v}(M)_{\mathcal{R}_i} = \overrightarrow{v}(O_m)_{\mathcal{R}i} + \overrightarrow{\omega}_{\mathcal{R}_m/\mathcal{R}_i} \wedge \overrightarrow{O_mM}.$$

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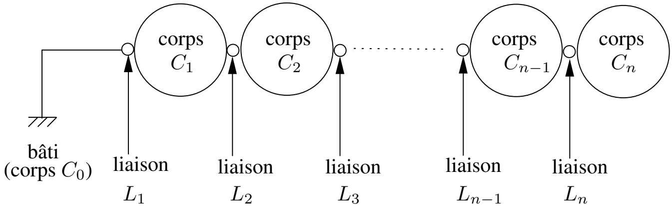
Description of Robotic manipulator

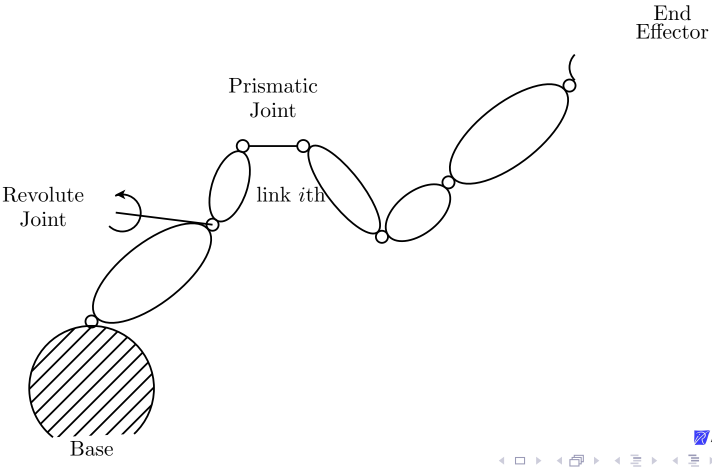
Definition

Robotic manipulator : n moving rigid bodies coupled by n revolute or prismatic joints



Description of Robotic manipulator

Only revolute (rotation) or prismatic (translation) joints.



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Modified Denavit-Hartenberg Method

Notations

- Each joint and link are numbered from the base to the end effector: the base body is link 0, the end effector is the n-th link.
- i-th joint and i-th link : frame $\{\mathcal{F}_i\} = (O_i, x_i, y_i, z_i)$, with $i = 0, 1, \ldots n$.
- Point O_{n+1} : attached to the end effector.
- Frame $\{\mathcal{F}_n\} = (O_n, x_n, z_n)$: such that $O_{n+1} \in \{\mathcal{F}_n\}$.

Modified Denavit-Hartenberg Method

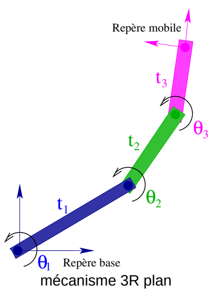
The general working plan is:

- What are the degrees of freedom of the manipulator ?
- Identify the joint coordinates.
- Identify the geometric parameters that define the manipulator.
- Associate to each joint a frame : from the base (0) to the end effector (n).
- Determine the position (matrix R, vector P) of each frame w.r.t the previous one: start from the end effector ($\mathcal{F}_n$) to the base ($\mathcal{F}_0$).
- Find the homogeneous transformation associated to each change of frame:

$$T = T_0^1 T_1^2 \dots T_{n-1}^n$$

Modified Denavit-Hartenberg Method

Example

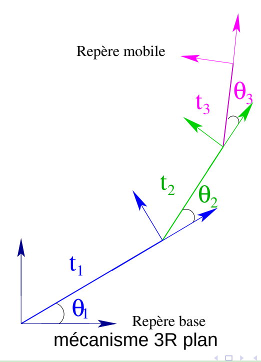


RRR planar Robot.

$$q = (\theta_1, \theta_2, \theta_3)^T, \zeta = (t_1, t_2, t_3)^T$$

Modified Denavit-Hartenberg Method

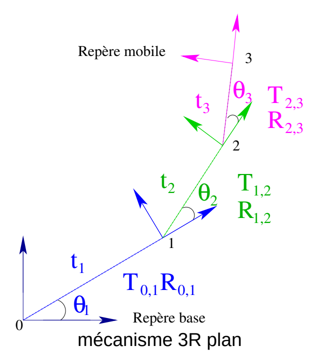
Example (continue)



Frames.

Modified Denavit-Hartenberg Method

Example (continue)



Frames.

For $i = 0, 1, 2$ and $j = 1, 2, 3$ we have

$$R_i^j = \begin{pmatrix} c_j & -s_j \\ s_j & c_j \end{pmatrix},$$

$$P_i^j = \begin{pmatrix} t_j c_j \\ t_j s_j \end{pmatrix},$$

$$T_i^j = \begin{pmatrix} R_i^j & T_i^j \\ (0)_{(2\times 2)} & 1 \end{pmatrix}$$

Modified Denavit-Hartenberg Method

Example (continue)

$$T_0^3 = \begin{pmatrix} \mathrm{c}_1 & -\mathrm{s}_1 & t_1\mathrm{c}_1 \\ \mathrm{s}_1 & \mathrm{c}_1 & t_1\mathrm{s}_1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathrm{c}_2 & -\mathrm{s}_2 & t_2\mathrm{c}_2 \\ \mathrm{s}_2 & \mathrm{c}_2 & t_2\mathrm{s}_2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathrm{c}_3 & -\mathrm{s}_3 & t_3\mathrm{c}_3 \\ \mathrm{s}_3 & \mathrm{c}_3 & t_3\mathrm{s}_3 \\ 0 & 0 & 1 \end{pmatrix}$$

$$T_0^3 = \begin{pmatrix} \mathrm{c}_{123} & -\mathrm{s}_{123} & t_1\mathrm{c}_1 + t_2\mathrm{c}_{12} + t_3\mathrm{c}_{123} \\ \mathrm{s}_{123} & \mathrm{c}_{123} & t_1\mathrm{s}_1 t_2\mathrm{s}_{12} t_3\mathrm{s}_{123} \\ 0 & 0 & 1 \end{pmatrix}$$

Modified Denavit-Hartenberg Method

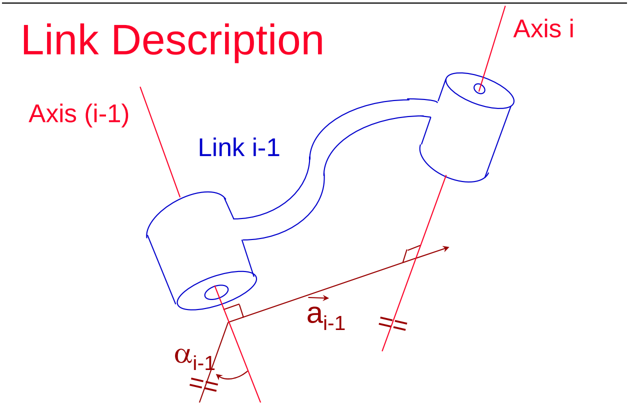
Formalism (Khalil 96):

- the i-th joint J_i is a perfect revolute or prismatic joint, this is with a single axis, thus represented by a single parameter.
- $\{\mathcal{F}_i\} = (O_i, x_i, y_i, z_i)$ is the frame associated to the i-th joint.
- O_{i-1} is the point of J_{i-1} on the common normal to J_{i-1} and J_i (in french "pied de la perpendiculaire commune"). In the case of parallel axis, arbitrary choice of the perpendicular line.

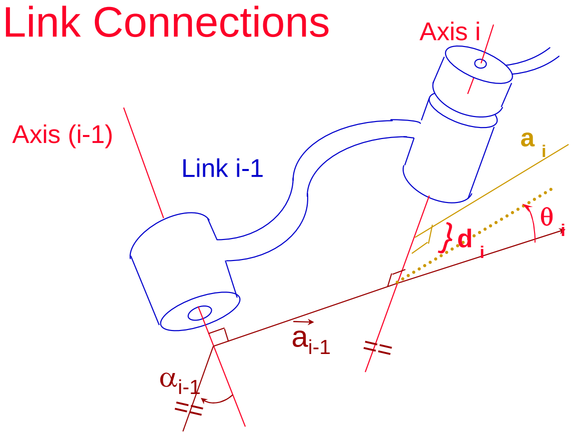
Modified Denavit-Hartenberg Method

- x_{i-1} is the unit vector of this common normal oriented from J_{i-1} to J_i. Arbitrary orientation when J_{i-1} and J_i intersect or when $J_{i-1} = J_i$ (up, right or front).
- z_{i-1} is the unit vector of the axis of the joint J_{i-1}, with an arbitrary orientation (up, right or front).
- y_{i-1} cross product of x_{i-1} and z_{i-1}: such that $\{\mathcal{F}_{i-1}\}$ is a right-hand frame.
- For $i = 0$, z_0 is up (in french "verticalement ascendant") and x_0 is orthogonal to the J_1 axis.
- For $i = n$, O_n is on the axe L_n and z_n is carried by the axis of the link n.

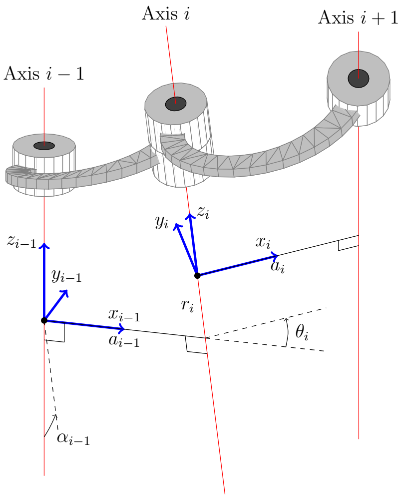
Modified Denavit-Hartenberg Method



Modified Denavit-Hartenberg Method



Modified Denavit-Hartenberg Method



Video

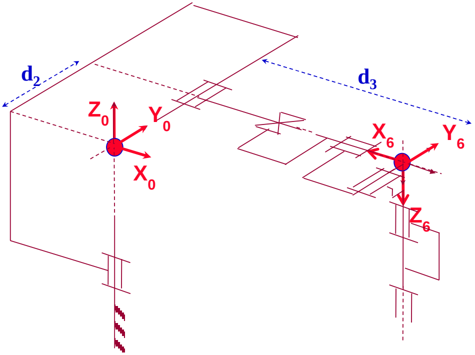
Modified Denavit-Hartenberg Method

Denavit-Hartenberg Parameters of the i-th link :

$$(\alpha_i, a_i, r_i, \theta_i)$$

- 3 fixed link parameters
- 1 joint variable : revolute (θ_i), prismatic (r_i)
- α_i, a_i describe the i-th link (geometry)
- r_i, θ_i describe the i-th link's connection

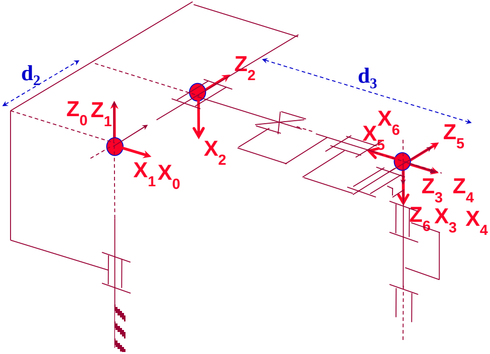
Description of Robotic manipulator



Stanford Scheinman Arm.

First and last frames of the base and the last joint before the end effector

Description of Robotic manipulator



Stanford Scheinman Arm.

Description of Robotic manipulator

Finally, the DH parameter of a manipulator are given in a table as follows:

Joint	α_{i-1}	a_{i-1}	r_i	θ_i
1	0	0	0	θ_1
2	-90	0	r_2	θ_2
3	90	0	r_3	0
4	0	0	0	θ_4
5	-90	0	0	θ_5
6	90	0	0	θ_6

Modified Denavit-Hartenberg Method

$$T_{i-1}^{i} = \mathrm{Rot}(\alpha_{i-1}, x_{i-1}).\mathrm{Trans}(a_{i-1}, x_{i-1}).\mathrm{Rot}(\theta_i, z_i).\mathrm{Trans}(r_i, z_i)$$

$$\mathrm{Rot}(\alpha_{i-1}, x_{i-1}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\alpha_{i-1}) & -\sin(\alpha_{i-1}) & 0 \\ 0 & \sin(\alpha_{i-1}) & \cos(\alpha_{i-1}) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{5}$$

$$\mathrm{Trans}(a_{i-1}, x_{i-1}) = \begin{pmatrix} 1 & 0 & 0 & a_{i-1} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{6}$$

$$\mathrm{Rot}(\theta_i, z_i) = \begin{pmatrix} \cos(\theta_i) & -\sin(\theta_i) & 0 & 0 \\ \sin(\theta_i) & \cos(\theta_i) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{7}$$

$$\mathrm{Trans}(r_i, z_i) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & r_i \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{8}$$

Description of Robotic manipulator

$$T_{i-1}^{i} = \begin{pmatrix} \mathrm{c}(\theta_i) & -\mathrm{s}(\theta_i) & 0 & a_{i-1} \\ \mathrm{c}(\alpha_{i-1})\mathrm{s}(\theta_i) & \mathrm{c}(\alpha_{i-1})\mathrm{c}(\theta_i) & -\mathrm{s}(\alpha_{i-1}) & -r_i\mathrm{s}(\alpha_{i-1}) \\ \mathrm{s}(\alpha_{i-1})\mathrm{s}(\theta_i) & \mathrm{s}(\alpha_{i-1})\mathrm{c}(\theta_i) & \mathrm{c}(\alpha_{i-1}) & r_i\mathrm{c}(\alpha_{i-1}) \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{9}$$

Description of Robotic manipulator

Stanford Scheinman Arm

$$T_0^1 = \begin{pmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{10}$$

$$T_1^2 = \begin{pmatrix} c_2 & -s_2 & 0 & 0 \\ 0 & 0 & 1 & r_2 \\ -s_2 & c_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{11}$$

Description of Robotic manipulator

Stanford Scheinman Arm

$$T_2^3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -r_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (12)$$

$$T_3^4 = \begin{pmatrix} \mathrm{c}_4 & -\mathrm{s}_4 & 0 & 0 \\ \mathrm{s}_4 & \mathrm{c}_4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (13)$$

Description of Robotic manipulator

Stanford Scheinman Arm

$$T_4^5 = \begin{pmatrix} c_5 & -s_5 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -s_5 & -c_5 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{14}$$

$$T_5^6 = \begin{pmatrix} c_6 & -s_6 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -s_5 & -c_5 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{15}$$

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Kinematic and Dynamic model

Configuration

Definition

Configuration of a mechanical system : set of minimal number of parameters, that give the position of any point of the system in a given frame.

Robotic manipulators case

Configuration of a robotic manipulator : vector q of n independent coordinates called the generalized coordinates. The set of admissible generalized coordinates is the configuration space $\mathcal{M}_{\mathrm{II}}$.

Generalized coordinates : rotation angles for revolute joints, translation values for prismatic joints.

Kinematic and Dynamic model

Pose (Posture)

Definition

Pose (or posture) of a rigid body : position and orientation of this rigid body in a given frame.

Robotic manipulators case

Pose of the end effector of a robotic manipulator : vector x of m independent operational coordinates. The set of admissible operational coordinates is the operational space $\mathcal{M}_{\S}$, of dimension $m \leq 6$.

Depends on the task (planar cases, positioning only ...) and on the parameterization of orientations.

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Kinematic and Dynamic model

Direct kinematics Model

Definition

Direct kinematic model (DKM) of a robotic manipulator : pose of the end effector as a function of the robot configuration :

$$f : \mathcal{M}_q \rightarrow \mathcal{M}_x \quad (16)$$

$$q \mapsto x = f(q) \quad (17)$$

General case

Expression of $x = (x_1 x_2 x_3 x_4 x_5 x_6)^T$, with $(x_1 x_2 x_3)^T$ the position coordinates in the frame $\mathcal{F}_0$ (of the end effector) and $(x_4 x_5 x_6)^T$ orientation coordinates, as a function of $q = (q_1 q_2 \ldots q_n)^T$.
Often using partial direction cosines.

Kinematic and Dynamic model

DKM computation

Orientation computed from the rotation matrix between the ground frame and the end effector frame.

Position $(x_1 x_2 x_3)^T$ of point O_{n+1} computed from the position $(p_x p_y p_z)^T$ of point O_n in the frame $\mathcal{F}_0$, given the coordinates $(a_n 0 r_{n+1})^T$ of O_{n+1} in the frame $\mathcal{F}_n$:

$$x_1 = p_x + a_n x_x + r_{n+1} z_x \quad (18)$$

$$x_2 = p_y + a_n x_y + r_{n+1} z_y \quad (19)$$

$$x_3 = p_z + a_n x_z + r_{n+1} z_z \quad (20)$$

Kinematic and Dynamic model

Practical considerations

Computation of On coordinates and of the direction cosines :

$$T_0^n(q) = T_0^1(q_1)T_1^2(q_2)\dots T_{n-1}^n(q_n)$$

Rules:

- For $i, j, \dots \in \{1, 2, \dots n\}$, write (for the sake of simplicity) :

  $$\mathrm{s}_i = \sin q_i, \mathrm{c}_i = \cos q_i, \mathrm{s}_{ij} = \sin(q_i + q_j), \mathrm{c}_{ij} = \cos(q_i + q_j)$$

  For example s_{123} means $\mathrm{s}_{123} = \sin(q_1 + q_2 + q_3)$.
- Each new mathematical operation (addition, multiplication) : introduction of a new intermediate variable.
- Reverse product computation : no computation of the second column of the rotation matrix.
- First compose transformations with particular properties, i.e. particular rotations with parallel axis.

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Kinematic and Dynamic model

Inverse Kinematics Model

Definition

Inverse kinematic model (IKM) : the (one or several) configurations that correspond to a given pose of the robot end effector :

$$\begin{aligned} f^{-1} : \mathcal{M}_x &\to \mathcal{M}_q && (21) \\ x &\mapsto q = f^{-1}(x) && (22) \end{aligned}$$

(Be careful, such x is not unique (generally, since f is not surjective).

Kinematic and Dynamic model

Inverse Kinematics Model

Solvability

Existence of solutions.

- If $n < m$: no solution.
- If $n = m$: finite number of solutions (in general).
- If $n > m$: infinite number of solutions.

Kinematic and Dynamic model

IKM computation

☞ There is no systematic analytic method.

☞ We obtain a set of nonlinear equations formed by the DKM $\Rightarrow$ Have a look at Algebraic geometry (transform such equations into a set of polynomial ones: look at Grobner Basis to find a generator of the ideal that generates the solutions.

Kinematic and Dynamic model

IKM computation

☞ When $n = 6$, the system is generally equipped with a spherical wrist that allows some decoupling between orientation and position, which helps solving the IKM :

$$p_x = x_1 - a_n x_x - r_{n+1} z_x \tag{23}$$
$$p_y = x - 2 + a_n x_y - r_{n+1} z_y \tag{24}$$
$$p_z = x_3 + a_n x_z - r_{n+1} z_z \tag{25}$$

Unfortunately the next steps require much more complex computations to solve for the q_i, $i \in \{1, 2, \ldots n\}$ as functions of the obtained p_x, p_y, p_z and of the direction cosines.

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Kinematic and Dynamic model

Direct Differential Kinematics Model

Definition

Direct differential kinematic model (DDKM) : relation between the operational velocities $\dot{x}$ and the generalized velocities $\dot{q}$:

$$\dot{x} = J(q)\dot{q}$$

where $J = J(q)$ is the Jacobian matrix of function f (f DKM $x = f(q)$), of dimension $m \times n$:

$$\begin{aligned} J : T_q\mathcal{M}_q &\rightarrow T_x\mathcal{M}_x && (26) \\ \dot{q} &\mapsto \dot{x} = J\dot{q} && (27) \end{aligned}$$

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Kinematic and Dynamic model

Mecanical Background

Fact

There are several ways to obtain and present the main results of mechanical engineering. Here choose to briefly recall the fundamental principle of dynamics and the Euler-Lagrange's formulation.

Kinematic and Dynamic model

Mecanical Background

ICI à compléter Quelques rappels sur masse, centre de centre de gravité (de masse) Théorème de Guldin et calcul de matrices d'inertie Théorème huygens etc... (cf. cours de mécanique)

Kinematic and Dynamic model

Mecanical Background: torsor

Definition

A **(in french "torseur"** denoted by

$$\mathcal{T} = \left\{ \begin{array}{c} \overrightarrow{x} \\ \overrightarrow{\mathcal{M}}(\overrightarrow{x})_{/O} \end{array} \right\}$$

is a pair of objects consisting of a vector noted $\overrightarrow{x}$ and a resultant moment at a point O of this vector noted $\overrightarrow{\mathcal{M}}(\overrightarrow{x})_{/O}$ both of them being linked by the following relation

$$\overrightarrow{\mathcal{M}}(\overrightarrow{x})_{/M} = \overrightarrow{\mathcal{M}}(\overrightarrow{x})_{/O} + \overrightarrow{x} \wedge \overrightarrow{OM}.$$

Kinematic and Dynamic model

Mecanical Background : torseur cinématique

Definition

En particulier pour le **torseur cinématique** dit aussi torseur des vitesses $\mathcal{T}_c = \left\{ \begin{array}{c} \overrightarrow{\omega} \\ \overrightarrow{v}(O) \end{array} \right\}$

$$\overrightarrow{v}(M) = \overrightarrow{v}(O) + \overrightarrow{\omega} \wedge \overrightarrow{OM}$$

Kinematic and Dynamic model

Mecanical Background : torseur cinématique

Definition

Torseur cinétique ou des quantités de mouvement

$$\mathcal{T}_p = \left\{ \begin{array}{c} \overrightarrow{p} = \sum_i m_i \overrightarrow{v}_i = m\overrightarrow{v}_G \\ \overrightarrow{\sigma}(O) = \sum_i \overrightarrow{OM_i} \wedge m_i \overrightarrow{v}_i \end{array} \right\} \text{avec}$$

$$\begin{aligned} \overrightarrow{\sigma}(O) &= \overrightarrow{\sigma}(G) + \overrightarrow{p} \wedge \overrightarrow{GO} \\ &= \overrightarrow{\sigma}(G) + \overrightarrow{OG} \wedge \overrightarrow{p} \end{aligned}$$

G est le centre de gravité il est définit par $\sum_i m_i \overrightarrow{OG} = \sum_i m_i \overrightarrow{OM_i}$; enfin $\overrightarrow{\sigma}(G) = J\overrightarrow{\omega}$ avec J la matrice d'inertie ou tenseur d'inertie dit encore matrice des moments d'inertie.

Kinematic and Dynamic model

Mecanical Background : torseur des quantités d'accélération

Definition

Le **torseur dynamique** dit aussi **torseur des quantités d'accélération** $\mathcal{T}_d = \left\{ \begin{array}{c} \sum_i m_i \overrightarrow{a}_i = m\overrightarrow{a}_G \\ \overrightarrow{\mu}(O) = \sum_i \overrightarrow{OM_i} \wedge m_i \overrightarrow{a}_i \end{array} \right\}$ et torseur des forces extérieurs $\mathcal{T}_{f_{\text{ext}}} = \left\{ \begin{array}{c} \sum_i \overrightarrow{F}_i \\ \sum_i \overrightarrow{\mathcal{M}}(\overrightarrow{F}_i)_{/O} \end{array} \right\}$.

Le principe fondamentale de la dynamique PFD s'exprime par

$$\mathcal{T}_d = \mathcal{T}_{f_{\text{ext}}}.$$

Kinematic and Dynamic model

Mecanical Background : Mouvement du point

Fact

Puisque $\mathcal{T}_{f_{ext}}$ *ne comporte qu'une composante, le PFD devient*

$$\sum \overrightarrow{F} = \frac{d\overrightarrow{p}}{dt} = m\overrightarrow{a}.$$

Kinematic and Dynamic model

Mecanical Background : Mouvement du solide

Fact

$\mathcal{T}_{f_{ext}}$ *comporte deux composantes :*

1. *la somme des forces extérieurs (donne lieu à un mouvement en translation)*
2. *la somme des moments (donne lieu à un mouvement en rotation).*

Kinematic and Dynamic model

Mecanical Background : translation

Fact

Translation :

$$\sum \overrightarrow{F} = \frac{d\overrightarrow{p}_G}{dt} = m\overrightarrow{a}_G,$$

si la masse est constante et G *est le centre de gravité*

Kinematic and Dynamic model

Mecanical Background : rotation

Fact

Rotation :

$$\sum \overrightarrow{\mathcal{M}}(\overrightarrow{F})_{/O} = \frac{d\overrightarrow{\sigma_O}}{dt},$$

où $\overrightarrow{\mathcal{M}}(\overrightarrow{F})_{/O}$ est le moment de la force par rapport au point O : $\overrightarrow{\mathcal{M}}(\overrightarrow{F})_{/O} = \overrightarrow{OA} \wedge \overrightarrow{F}$ (A étant le point d'application de la force $\overrightarrow{F}$) et $\overrightarrow{\sigma_O}$ le moment cinétique : $\overrightarrow{\sigma_O} = \overrightarrow{OG} \wedge \overrightarrow{p}$.

Kinematic and Dynamic model

Mecanical Background : Principe de l'action et de la réaction

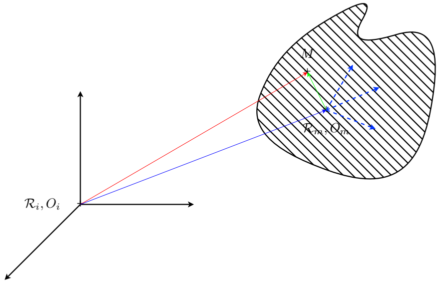
Fact

Principe de l'action et de la réaction :

$$\overrightarrow{f_{12}} + \overrightarrow{f_{21}} = \overrightarrow{0}.$$

Kinematic and Dynamic model

Mecanical Background : Example



Solide dans l'espace (satellite) : changement de repère.

Kinematic and Dynamic model

Mecanical Background : Example

Example (Satellite en rotation)

le PFD s'écrit

$$\sum \overrightarrow{\mathcal{M}}(\overrightarrow{F})_{O_m}\Big|_{\mathcal{R}_i} = \frac{d\overrightarrow{\sigma}_{O_m}}{dt}\Big|_{\mathcal{R}_i}$$

avec ici $\overrightarrow{\sigma}_{O_m} = \int_{\text{solide}} \overrightarrow{O_mM} \wedge \overrightarrow{v}(M)_{\mathcal{R}_i}$

Principes de la physique

Mécanique

Example (Satellite en rotation)

Puisque $\overrightarrow{v}(M)_{\mathcal{R}_i} = \overrightarrow{v}(O_m)_{\mathcal{R}_i} + \overrightarrow{\omega}_{\mathcal{R}_m/\mathcal{R}_i} \wedge \overrightarrow{O_m M}$,

$$\begin{aligned}
\overrightarrow{\sigma}_{O_m} &= \int_{\text{solide}} \overrightarrow{O_m M} \wedge \overrightarrow{v}(M)_{\mathcal{R}_i} dm \\
&= \int_{\text{solide}} \overrightarrow{O_m M} \wedge \left[\overrightarrow{v}(O_m)_{\mathcal{R}_i} + \overrightarrow{\omega}_{\mathcal{R}_m/\mathcal{R}_i} \wedge \overrightarrow{O_m M} \right] dm \\
&= -\overrightarrow{v}(O_m)_{\mathcal{R}_i} \wedge \int_{\text{solide}} \overrightarrow{O_m M} dm + \\
& \quad \int_{\text{solide}} \overrightarrow{O_m M} \wedge \left(\overrightarrow{\omega}_{\mathcal{R}_m/\mathcal{R}_i} \wedge \overrightarrow{O_m M} \right) dm
\end{aligned}$$

Kinematic and Dynamic model

Mecanical Background : Example

Example (Satellite en rotation)

En choisissant O_m = centre de gravité $\int_{\text{solide}} \overrightarrow{O_m M} dm = \overrightarrow{0}$ et $\int_{\text{solide}} \overrightarrow{O_m M} \wedge \left(\overrightarrow{\omega}_{\mathcal{R}_m/\mathcal{R}_i} \wedge \overrightarrow{O_m M} \right) dm = I \overrightarrow{\omega}_{\mathcal{R}_m/\mathcal{R}_i}$ où I est la matrice d'inertie définie par

$$I = \begin{pmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{pmatrix},$$

avec $I_{xx} = \int_{\text{solide}} (y^2 + z^2) dm$, $I_{yy} = \int_{\text{solide}} (x^2 + z^2) dm$, $I_{zz} = \int_{\text{solide}} (x^2 + y^2) dm$, $I_{xy} = \int_{\text{solide}} xy dm$, $I_{xz} = \int_{\text{solide}} xz dm$, $I_{yz} = \int_{\text{solide}} yz dm$.

Kinematic and Dynamic model

Mecanical Background : Example

Example (Satellite en rotation)

Finalement

$$\overrightarrow{\sigma}_{O_m} = I\overrightarrow{\omega}_{\mathcal{R}_m/\mathcal{R}_i}$$

De même $\left.\frac{d\overrightarrow{\sigma}_{O_m}}{dt}\right|_{\mathcal{R}_i} = \left.\frac{d\overrightarrow{\sigma}_{O_m}}{dt}\right|_{\mathcal{R}_m} + \overrightarrow{\omega}_{\mathcal{R}_m/\mathcal{R}_i} \wedge \overrightarrow{\sigma}_{O_m}|_{\mathcal{R}_m}$

Kinematic and Dynamic model

Mecanical Background : Example

Example (Satellite en rotation)

Si les axes de $\mathcal{R}_m$ sont les axes principaux d'inertie $(I_{xy} = I_{xz} = I_{yz} = 0)$ on en déduit

$$\begin{pmatrix} I_{xx}\frac{d\omega_x}{dt} \\ I_{yy}\frac{d\omega_y}{dt} \\ I_{zz}\frac{d\omega_z}{dt} \end{pmatrix} = \begin{pmatrix} (I_{yy} - I_{zz})\omega_y\omega_z + \sigma_{O_m}|_x \\ (I_{zz} - I_{xx})\omega_x\omega_z + \sigma_{O_m}|_y \\ (I_{xx} - I_{yy})\omega_x\omega_y + \sigma_{O_m}|_z \end{pmatrix}$$

Kinematic and Dynamic model

Mecanical Background : Formalisme d'Euler-Lagrange

Fact

Si un système mécanique est constitué de n éléments reliés entre eux par des liaisons parfaites (rigides et sans frottement !), alors la position du système dépendra de n paramètres indépendants (coordonnées généralisées notées $q_1, \dots, q_n$).

Principes de la physique

Mécanique : Formalisme d'Euler-Lagrange

Pour écrire les équations d'Euler-Lagrange il faut déterminer le Lagrangien (différence entre l'énergie cinétique et l'énergie potentielle) :

$$\mathcal{L} = \mathcal{E}_c - \mathcal{E}_p, \tag{28}$$

le travail élémentaire de chaque forces interne et externe D_i, ainsi que le travail des forces de frottements :

$$-\frac{\partial D}{\partial \dot{q}_i} dq_i, \tag{29}$$

donnant lieu à l'énergie dissipe D. On obtient alors le système d'équations d'Euler-Lagrange :

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} + \frac{\partial D}{\partial \dot{q}_i} = D_i. \tag{30}$$

Principes de la physique

Mécanique : Formalisme d'Euler-Lagrange

Remark

$\mathcal{E}_c$ dépend des q_i et de leurs dérivées $\dot{q}_i$ alors que $\mathcal{E}_p$ ne dépend que des q_i.

Remark

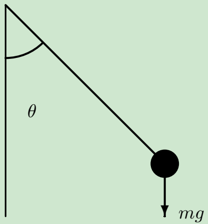
$\mathcal{E}_c = \frac{1}{2}\dot{q}^T M(q)\dot{q}$, où $M(q)$ est une matrice $n \times n$ symétrique définie positive.

Principes de la physique

Mécanique : Formalisme d'Euler-Lagrange : pendule pesant

Example (Pendule pesant)

Une masse m est attachée à un pivot sans frottement à l'aide d'un fil non pesant de longueur l.



Pendule pesant.

Principes de la physique

Mécanique : Formalisme d'Euler-Lagrange : pendule pesant

Example (Pendule pesant)

$$\mathcal{L} = \frac{1}{2} m l^2 \dot{\theta}^2 - mgl(1 - \cos(\theta)), D = 0.$$

En appliquant directement (30), on obtient :

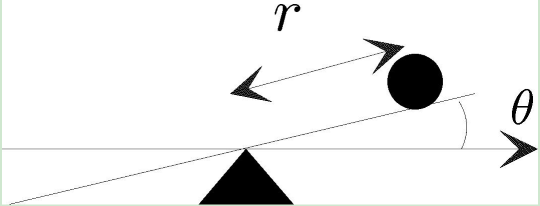
$$(l\ddot{\theta} + g\sin(\theta)) = 0.$$

Principes de la physique

Mécanique : Formalisme d'Euler-Lagrange : Bille sur un rail

Example (Bille sur un rail)

Une bille de masse m roule sur un rail incliné lui-même actionné par un couple u appliqué autour de l'axe de rotation du rail.



Bille sur un rail.

Principes de la physique

Mécanique : Formalisme d'Euler-Lagrange : Bille sur un rail

Example (Bille sur un rail)

Euler Lagrange: Paramètres (coordonnées généralisées notées $q_1, \ldots, q_n$): r, θ.

$$\begin{aligned}
\mathcal{L} &= \mathcal{E}_c - \mathcal{E}_p, \\
\mathcal{E}_c &= \frac{1}{2} m\dot{r}^2 + \frac{1}{2}(mr^2 + J)\dot{\theta}^2, \\
\mathcal{E}_p &= mgr\sin(\theta), \\
\mathcal{L} &= \frac{1}{2} m\dot{r}^2 + \frac{1}{2}(mr^2 + J)\dot{\theta}^2 - mgr\sin(\theta)
\end{aligned}$$

Principes de la physique

Mécanique : Formalisme d'Euler-Lagrange : Bille sur un rail

Example (Bille sur un rail)

Travail élémentaire de chaque forces interne et externe D_i, ainsi que le travail des forces de frottements $-\frac{\partial D}{\partial \dot{q}_i} dq_i$, donnant lieu à l'énergie dissipée D.

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} + \frac{\partial D}{\partial \dot{q}_i} &= D_i. \\
(mr^2 + J)\ddot{\theta} + 2mr\dot{r}\dot{\theta} + mgr\cos(\theta) &= u, \\
m\ddot{r} + mg\sin(\theta) - mr\dot{\theta}^2 &= 0.
\end{aligned}$$

Principes de la physique

Mécanique : Formalisme d'Euler-Lagrange : Bille sur un rail

Example (Bille sur un rail)

En posant $x = \begin{pmatrix} r & \dot{r} & \theta & \dot{\theta} \end{pmatrix}^{\mathsf{T}}$ et avec le retour statique $u = 2mr\dot{r}\dot{\theta} + mgr\cos(\theta) + (mr^2 + J)v$:

$$\dot{x} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \dot{\theta}^2 & 0 & -g\frac{\sin(\theta)}{\theta} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} v$$

Principes de la physique

Mécanique : Formalisme d'Euler-Lagrange : Bille sur un rail

Example (Bille sur un rail)

Linéarisé (avec pre-retour statique)

$$\left\{\begin{array}{l} \dot{x} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -g & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} v, \\ y = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix} x, \\ x = \begin{pmatrix} r & \dot{r} & \theta & \dot{\theta} \end{pmatrix}^{\mathsf{T}}. \end{array}\right. ,$$

Principes de la physique

Mécanique : Formalisme d'Euler-Lagrange : Bille sur un rail

Example (Bille sur un rail)

Linéarisé (sans pre-retour static)

$$\begin{aligned} (mr^2 + J)\ddot{\theta} + mgr\cos(\theta) &= u, \\ \ddot{r} + g\theta &= 0. \end{aligned}$$

$$\left\{ \begin{array}{l} \dot{x} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -g & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{mg}{J} & 0 & 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{J} \end{pmatrix} u, \\ y = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix} x, \\ x = \begin{pmatrix} r & \dot{r} & \theta & \dot{\theta} \end{pmatrix}^{\mathsf{T}}. \end{array} \right. ,$$

Kinematic and Dynamic model

Dynamic model: Euler-Lagrange framework

Manipulator: The total kinetic energy is the sum of the kinetic energy pour each part (n joint, links) + the kinetic energy for the actuators.

☞ Kinetic Energy for i-th part: Use of DH (Denavit-Hartenberg) parametrization.

$$v_{i/0} = \frac{dr_{i/0}}{dt} \tag{31}$$

$$r_{i/0} = T_0^i r_{i/i} \tag{32}$$

Kinematic and Dynamic model

Dynamic model: Euler-Lagrange framework

Rigid body : $\frac{dr_{i/i}}{dt} = 0$

$$v_0^i = \sum_{j=1}^{i} \frac{\partial T_0^i}{\partial q_j} \frac{dq_j}{dt} r_{i/i} \tag{33}$$

$$= \sum_{j=1}^{i} U_{ij} \dot{q}_j r_{i/i} \tag{34}$$

where

$$U_{ij} = \begin{cases} T_0^{j-1} Q_j T_{j-1}^i, & j \leq i \\ 0, & j > i \end{cases} \tag{35}$$

Kinematic and Dynamic model

Dynamic model: Euler-Lagrange framework

For revolute joint (rotation):

$$Q_j = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{36}$$

For prismatic joint (translation):

$$Q_j = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{37}$$

Kinematic and Dynamic model

Dynamic model: Euler-Lagrange framework

Since

$$d\mathcal{E}_{ci} = \frac{1}{2}\mathrm{trace}(v_i v_i^T)dm \quad (38)$$

$$= \frac{1}{2}\mathrm{trace}\left(\sum_{j=1}^{i}\sum_{k=1}^{i}\right) U_{ij}(r_{i/i} r_{i/i}^T dm) U_{ik}^T \dot{q}_j \dot{q}_k \quad (39)$$

Letting:

$$J_i = \int r_{i/i} r_{i/i}^T dm$$

one gets:

Kinematic and Dynamic model

Dynamic model: Euler-Lagrange framework

$$\mathcal{E}_{ci} = \frac{1}{2}\text{trace}\left(\sum_{j=1}^{i}\sum_{k=1}^{i}\right) U_{ij} J_i U_{ik}^T \dot{q}_j \dot{q}_k$$

$$J_i = \begin{pmatrix} \int x_i^2 dm & \int x_i y_i dm & \int x_i z_i dm & \int x_i dm \\ \int x_i y_i dm & \int y_i^2 dm & \int y_i z_i dm & \int y_i dm \\ \int x_i z_i dm & \int y_i z_i dm & \int z_i^2 dm & \int z_i dm \\ \int x_i dm & \int y_i dm & \int z_i dm & \int dm \end{pmatrix} \tag{40}$$

Kinematic and Dynamic model

Dynamic model: Euler-Lagrange framework

The kinetic energy for the actuators is:

$$\mathcal{E}_{ca} = \frac{1}{2} \sum_{i=1}^{n} I_i \dot{q}_i^2$$

where I_i is an inertia momentum (for rotation) or a mass (for translation). This the total kinetic energy is

$$\mathcal{E}_c = \sum_{i=1}^{n} \mathcal{E}_{ci} + \mathcal{E}_{ca}$$

Kinematic and Dynamic model

Dynamic model: Euler-Lagrange framework

The potential energy is:

$$\mathcal{E}_p = \sum_{i=1}^{n} -m_i g^T r_{i/0}$$

$$g = (g_x g_y g_z 1)^T$$

(when g is on the z axis $g = (00 - g1)^T$.). Since the dissipative energy is $\mathcal{E}_D = \frac{1}{2}\sum_{i=1}^{n} f_{vi}\dot{q}_i^2$, thus the Lagragian $\mathcal{L} = \mathcal{E}_c - \mathcal{E}_p$ is:

$$\mathcal{L} = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{k=1}^{i} \mathrm{trace}\left(U_{ij} J_i U_{ik}^T\right)\dot{q}_j\dot{q}_k + \sum_{i=1}^{n} m_i g^T r_{i/0}$$

Kinematic and Dynamic model

Dynamic model: Euler-Lagrange framework

Using relation (30), one gets

$$\Gamma_i = \sum_{j=1}^{n}\sum_{k=1}^{j} \operatorname{trace}\left(U_{jk}J_jU_{ji}^T\right)\ddot{q}_k + \sum_{j=1}^{n}\sum_{k=1}^{j}\sum_{l=1}^{j} \operatorname{trace}\left(U_{jkl}J_jU_{ji}^T\right)\dot{q}_k\dot{q}_l + f_{vi}\dot{q}_i - \sum_{j=1}^{n} m_i g^T U_{ji} r_{j/j} \tag{41}$$

where Γ_i are the torques (forces) applied to the ith body and

$$U_{ijk} = \begin{cases} T_0^{k-1}Q_kT_{k-1}^{j-1}Q_jT_{j-1}^{i}, & k \le j \le i \\ T_0^{j-1}Q_jT_{j-1}^{k-1}Q_kT_{k-1}^{i}, & j \le k \le i \\ 0, \quad j < i < k \end{cases} \tag{42}$$

Kinematic and Dynamic model

Dynamic model: Euler-Lagrange framework

$$\Gamma = M(q)\ddot{q} + N(q,\dot{q}) + G(q) + H(\dot{q}) \tag{43}$$

$$M_{ij}(q) = \sum_{k=\max(i,j)}^{n} \operatorname{trace}\left(U_{kj} J_k U_{ki}^T\right) \tag{44}$$

$$N_{ijk}(q) = \sum_{l=\max(i,j,k)}^{n} \operatorname{trace}\left(U_{ljk} J_l U_{li}^T\right) \tag{45}$$

$$G_i(q) = -\sum_{j=i}^{n} m_j g^T U_{ji} r_{j/j} \tag{46}$$

$$H_i(\dot{q}_i) = f_{vi}\dot{q}_i \tag{47}$$

Kinematic and Dynamic model

Dynamic model: Euler-Lagrange framework

Enjoy !!