Event-based control of networks modeled by a class of infinite-dimensional systems

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Ph.D Defense

This work has been partially supported by the LabEx PERSYVAL-Lab (ANR-11-LABX-0025)
1D-Hyperbolic partial differential equations

Modeling of physical networks

- Hydraulic: Saint-Venant equations for open channels [Bastin, Coron, and d’Andréa-Novel; 2008];
- Road traffic [Coclite, Garavello and Piccoli; 2005];
- Data/communication: Packets flow on telecommunication networks [D’Apice, Manzo and Piccoli; 2006];
- …

[“Stability and Boundary Stabilization of 1-D Hyperbolic Systems”, Bastin and Coron; 2016].

Event-based boundary control of these applications

- To propose a framework for event-based control of hyperbolic systems.
  - A rigorous way to implement digitally continuous time controllers for hyperbolic systems.
  - To reduce control and communication constraints.
Outline

1. **Networks of conservation laws (Chapter 1)**
   - Fluid-flow modeling
   - ISS stability

2. **Event-based stabilization (Chapter 2)**
   - ISS static event-based stabilization

3. **EBC-Backstepping approach (Chapter 4)**

4. **Conclusion and Perspectives**
Fluid-flow modeling - communication networks

Figure: Example of a compartmental network.

- $\mathcal{I}_n$: set of the number of compartments, numbered from 1 to $n$.
- $\mathcal{D}_i \subset \mathcal{I}_n$: index set of downstream compartments connected directly to compartment $i$ (i.e. those compartments receiving flow from compartment $i$).
- $\mathcal{U}_i \subset \mathcal{I}_n$: index set of upstream compartments connected directly to compartment $i$ (i.e. those compartments sending flow to compartment $i$).

Index sets involved in the example:

- $\mathcal{I}_n = \{1, 2, 3, 4\}$, $\mathcal{U}_1 = \emptyset$, $\mathcal{U}_2 = \{1\}$, $\mathcal{U}_3 = \{1, 2\}$, $\mathcal{U}_4 = \{2, 3\}$
- $\mathcal{D}_1 = \{2, 3\}$, $\mathcal{D}_2 = \{3, 4\}$, $\mathcal{D}_3 = \{4\}$, $\mathcal{D}_4 = \emptyset$. 
Transmission lines

Transmission lines may be modeled by the following conservation laws [D’Apice, Manzo, Piccoli; 2008]:

\[ \partial_t \rho_{ij}(t, x) + \partial_x f_{ij}(\rho_{ij}(t, x)) = 0 \]

- \(\rho_{ij}(t, x)\) is the density of packets;
- \(f_{ij}(\rho_{ij}(t, x))\) is the flow of packets, \(x \in [0, 1], t \in \mathbb{R}^+, i \in \mathcal{I}_n, j \in \mathcal{D}_i\).

\[ f_{ij}(\rho_{ij}) = \begin{cases} 
\lambda_{ij} \rho_{ij}, & \text{if } 0 \leq \rho_{ij} \leq \sigma_{ij} \\
\lambda_{ij}(2\sigma_{ij} - \rho_{ij}), & \text{if } \sigma_{ij} \leq \rho_{ij} \leq \rho_{ij}^{\max} 
\end{cases} \]

- \(\sigma_{ij}\) is the critical density - free flow zone and congested zone.
- \(\lambda_{ij}\) is the average velocity of packets traveling through the transmission line.

**Figure:** Fundamental triangular diagram of flow-density
We focus on the case in which the network operates in \textbf{free-flow}, i.e.

\[ f_{ij}(\rho_{ij}) = \lambda_{ij} \cdot \rho_{ij} \]

for \(0 \leq \rho_{ij} \leq \sigma_{ij}\).

Let us denote the flow \( f_{ij}(\rho_{ij}) := q_{ij} \).

We rewrite the conservation laws as \textit{Kinematic wave equation} [Bastin, Coron, d’Andréa-Novel; 2008]:

\[ \partial_t q_{ij}(t, x) + \lambda_{ij} \partial_x q_{ij}(t, x) = 0 \]
 Servers: Buffers and routers

Dynamics for each buffer \( i \in \mathcal{I}_n \) (see e.g. Congestion control in compartmental network systems [Bastin, Guffens; 2006]):

\[
\dot{z}_i(t) = v_i(t) - r_i(z_i(t))
\]

- \( v_i \) is the sum of all input flows getting into the buffer;
- \( r_i \) is the output flow of the buffer (processing rate function).

with \( v_i(t) = d_i(t) + \sum_{k \in \mathcal{U}_i, k \neq i} q_{ki}(t, 1) \).
The output flow $r_i(z_i)$ (processing rate function):

$$r_i(z_i) = \frac{z_i}{\theta_i(z_i)}$$

The *residence time* is the averaged time at which packets stay in the server when being processed.

$$\theta_i(z_i) = \frac{1 + z_i}{\epsilon_i}$$

with $\epsilon_i > 0$ as the maximal processing capacity of each server.
Control functions and dynamic boundary condition

Control functions

- $w_i$: To modulate the input flow $v_i$ and reject information.
- $u_{ij}$: To split the flow through different lines.

$$\dot{z}_i(t) = w_i(t)d_i(t) + \sum_{\substack{k \neq i \\ k \in U_i}} w_i(t)q_{ki}(t, 1) - r_i(z_i(t)), \quad w_i(t) \in [0, 1]$$

Dynamic boundary condition

$$q_{ij}(t, 0) = u_{ij}(t)r_i(z_i(t))$$

Splitting control (routing control): $u_{ij}(t) \in [0, 1], j \in D_i, i \in I_n$.

The output function for each output compartment $i \in I_{out}$ is given by

$$e_i(t) = u_i(t)r_i(z_i(t))$$

with $\sum_{\substack{i \neq j \\ j \in D_i}} u_{ij}(t) + u_i(t) = 1$. 
The complete model for the network is:

\[
\begin{cases}
\partial_t q_{ij}(t, x) + \lambda_{ij} \partial_x q_{ij}(t, x) = 0 \\
\dot{z}_i(t) = w_i(t)d_i(t) + \sum_{k \notin i} w_i(t)q_{ki}(t, 1) - r_i(z_i(t))
\end{cases}
\]  

(1)

with dynamic boundary condition

\[q_{ij}(t, 0) = u_{ij}(t)r_i(z_i(t)), \quad r_i \geq 0\]  

(2)

output function

\[e_i(t) = u_i(t)r_i(z_i(t))\]  

(3)

and initial conditions

\[
\begin{cases}
q_{ij}(0, x) = q_{ij}^0(x), \quad x \in [0, 1] \\
z_i(0) = z_i^0.
\end{cases}
\]  

(4)
Free-flow steady-state

For a given constant input flow demand $d_i^*$,

- the system (1)-(4) has infinitely many equilibrium points $\{q_{k_i}^*, z_i^*, u_{ij}^*, u_i^*, w_i^*, e_i^*\}$

$$
\begin{align*}
  w_i^* d_i^* + \sum_{k \neq i} \sum_{k \in U_i} w_i^* q_{k_i}^* - r_i(z_i^*) &= 0 \\
  q_{ij}^* &= u_{ij}^* r_i(z_i^*) \\
  e_i^* &= u_i^* r_i(z_i^*)
\end{align*}
$$

- We assume that the system admits a free-flow steady-state;
- Multi-objective optimization problem.

  - Maximizing the total output flow rate of the network;
  - Minimizing the total mean travel time.
Linearized system around an optimal free-flow equilibrium point

Coupled linear hyperbolic PDE-ODE.

\[
\begin{align*}
\partial_t y(t, x) + \Lambda \partial_x y(t, x) &= 0 \\
\dot{Z}(t) &= AZ(t) + G_y y(t, 1) + B_w W(t) + D \tilde{d}(t)
\end{align*}
\]

with dynamic boundary condition (B.C)
\[y(t, 0) = G_z Z(t) + B_u U(t)\]
and initial condition (I.C)
\[y(0, x) = y^0(x), \quad x \in [0, 1] \]
\[Z(0) = Z^0.\]
ISS properties for Hyperbolic systems

- [Prieur, Mazenc; 2012];
- [Prieur, Winkin, Bastin; 2008];
- [Tanwani, Prieur, Tarbouriech; CDC 2016].

**Definition (Input-to-State stability ISS)**

The system $\mathcal{P}$ is **Input-to-State Stable (ISS)** with respect to $\tilde{d} \in C_{pw}(\mathbb{R}^+; \mathbb{R}^n)$, if there exist $\nu > 0$, $C_1 > 0$ and $C_2 > 0$ such that, for every $Z^0 \in \mathbb{R}^n$, $y^0 \in L^2([0, 1]; \mathbb{R}^m)$, the solution satisfies, for all $t \in \mathbb{R}^+$,

$$
\left( \|Z(t)\|^2 + \|y(t, \cdot)\|_{L^2([0,1], \mathbb{R}^m)}^2 \right) \leq C_1 e^{-2\nu t} \left( \|Z^0\|^2 + \|y^0\|_{L^2([0,1], \mathbb{R}^m)}^2 \right) + C_2 \sup_{0 \leq s \leq t} \|\tilde{d}(s)\|^2 \tag{5}
$$

$C_2$ is called the **asymptotic gain (A.g)**.
Contributions on this framework:
- Modeling of communication networks under fluid-flow and compartmental representation;
- Characterization of suitable operating points for the network;

Open-loop analysis (Lyapunov-based):
- Sufficient condition for ISS - LMI formulation;
- Asymptotic gain estimation;

Closed-loop analysis (Lyapunov-based):
- Control synthesis to improve the performance of the network;
- LMI formulation;
- Control constraints;
- Minimization of the Asymptotic gain;

\[
W(t) = [K_z \quad K_y] \begin{bmatrix} Z(t) \\ y(t,1) \end{bmatrix} \\
U(t) = [L_z \quad L_y] \begin{bmatrix} Z(t) \\ y(t,1) \end{bmatrix}
\]
Contributions on this framework:
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Closed-loop analysis (Lyapunov-based):
- Control synthesis to improve the performance of the network;
- LMI formulation;
- Control constraints.
- Minimization of the Asymptotic gain;

\[ W(t) = \begin{bmatrix} K_z & K_y \end{bmatrix} \begin{bmatrix} Z(t) \\ y(t,1) \end{bmatrix} \]

\[ U(t) = \begin{bmatrix} L_z & L_y \end{bmatrix} \begin{bmatrix} Z(t) \\ y(t,1) \end{bmatrix} \]
Closed-loop setting objective:

- Reduce the asymptotic gain.

Therefore, the linearized coupled PDE-ODE system in closed-loop with $C$ becomes:

$$
\begin{align*}
\left\{ \begin{array}{l}
\partial_t y(t, x) + \Lambda \partial_x y(t, x) = 0 \\
\dot{Z}(t) = (A + B_w K_z)Z(t) + (G_y + B_w K_y)y(t, 1) + D \tilde{d}(t)
\end{array} \right. \\
\text{B.C:} \\
y(t, 0) = (G_z + B_u L_z)Z(t) + B_u L_y y(t, 1) \\
\text{I.C:} \\
y(0, x) = y^0(x) \\
Z(0) = Z^0
\end{align*}
$$
Theorem (Control synthesis)

Let \( \lambda = \min\{\lambda_{ij}\}_{i \in I, j \in D} \). Assume that there exist \( \mu, \gamma > 0 \), a symmetric positive definite matrix \( P \in \mathbb{R}^{n \times n} \) a diagonal positive matrix \( Q \geq I \in \mathbb{R}^{m \times m} \), as well as control gains \( K_z, K_y, L_z \) and \( L_y \) of adequate dimensions, such that the following matrix inequality holds:

\[
M_c = \begin{pmatrix}
M_1 & M_2 & M_3 \\
* & M_4 & 0 \\
* & * & M_5
\end{pmatrix} \leq 0
\]

with

- \( M_1 = (A + B_w K_z)^T P + P (A + B_w K_z) + 2 \mu \lambda P + (G_z + B_u L_z)^T Q \Lambda (G_z + B_u L_z) \);
- \( M_2 = P (G_y + B_w K_y) + (G_z + B_u L_z)^T Q \Lambda B_u L_y \);
- \( M_3 = PD \);
- \( M_4 = -e^{-2 \mu Q \Lambda} + L_y^T B_u^T Q \Lambda B_u L_y \);
- \( M_5 = -\gamma I \).

Then, the closed-loop system \( \mathcal{P} \) is ISS with respect to \( \tilde{d} \in C_{pw}(\mathbb{R}^+; \mathbb{R}^n) \), and the asymptotic gain (A.g) satisfies

\[
A.g \leq \frac{\gamma}{2 \mu \lambda} e^{2 \mu}.
\]
Optimization issues and control constraints

\[ \text{minimize} \quad \frac{\gamma}{2\nu} e^{2\mu} \]

\[ \text{subject to} \quad M_c \leq 0; \]

\[ \|K_{zi}\| \leq \frac{p\delta^w_i}{\beta_z}; \|K_{yi}\| \leq \frac{(1 - p)\delta^w_i}{\beta_y}; \|L_{zi}\| \leq \frac{\delta^u_{ij}}{\beta_z} \]
Running example

Network of compartments made up of 4 buffers and 5 transmission lines.

Figure: Network of compartments made up of 4 buffers and 5 transmission lines.
Exogenous input flow demand

Total output flow of the network

- **In open loop (black line)** - Asymptotic gain: 40.48
- **In closed loop (red dashed line)** - Asymptotic gain: 3.3
Event-based control (Chapter 2)
EBC of Linear hyperbolic system of conservation laws

\[ \frac{\partial}{\partial t} y(t, x) + \Lambda \frac{\partial}{\partial x} y(t, x) = 0 \]
\[ y(t, 0) = H y(t, 1) + B u(t) \]

Contributions on this framework:

- Event-triggered mechanisms (ETM);
  \[ t_{k+1} = \inf\{ t \in \mathbb{R}^+ | t > t_k \land \text{some Lyapunov-based triggering condition} \} \]
- ISS static event-based stabilization \( \phi_1 \);
- \( D^+V \) event-based stabilization \( \phi_2 \);
- ISS dynamic event-based stabilization \( \phi_3 \);
ASSUMPTION [A]

There exists a causal operator $\varphi$

$\varphi : \mathcal{C}_{pw}(\mathbb{R}^+, \mathbb{R}^n) \rightarrow \mathcal{C}_{pw}(\mathbb{R}^+, \mathbb{R}^m)$

$u = \varphi(y(\cdot, 1))$

$y^0 \in \mathcal{C}_{pw}([0, 1], \mathbb{R}^n), \ y(\cdot, 1) \in \mathcal{C}_{pw}(\mathbb{R}^+, \mathbb{R}^n)$

There exists a unique solution to the closed-loop system with controller $u = \varphi(y(\cdot, 1))$

Stability Analysis

$\varphi_1 :$ ISS static event-based stabilization

Let the control function be defined

$$
\begin{align*}
u(t) &= 0 \quad \forall t \in [t_0^u, t_1^u) \\
u(t) &= Ky(t_k^u, 1) \quad \forall t \in [t_k^u, t_{k+1}^u), \quad k \geq 1
\end{align*}
$$

Let us rewrite the boundary condition:

$$
\begin{align*}
y(t, 0) &= Hy(t, 1) + Bu(t) \\
&= Hy(t, 1) + BKy(t_k^u, 1) \\
&= (H + BK)y(t, 1) + BK(-y(t, 1) + y(t_k^u, 1)) \\
&= Gy(t, 1) + d(t)
\end{align*}
$$

- $d$ is seen as a deviation between the continuous controller $u = Ky(t, 1)$ and the event based controller (6).
- Seek for ISS property with respect to $d$!
\[ \varphi_1: \text{ISS static event-based stabilization} \]

Let the control function be defined

\[
\begin{align*}
    u(t) & = 0 & \forall t \in [t_0^u, t_1^u) \\
    u(t) & = Ky(t_k^u, 1) & \forall t \in [t_k^u, t_{k+1}^u), \quad k \geq 1
\end{align*}
\]

Let us rewrite the boundary condition:

\[
\begin{align*}
y(t, 0) & = Hy(t, 1) + Bu(t) \\
        & = Hy(t, 1) + BKy(t_k^u, 1) \\
        & = (H + BK)y(t, 1) + BK(-y(t, 1) + y(t_k^u, 1)) \\
        & = Gy(t, 1) + d(t)
\end{align*}
\]

- \( d \) is seen as a \textit{deviation} between the continuous controller \( u = Ky(t, 1) \) and the event based controller (6).

- Seek for ISS property with respect to \( d \)!
Recall:

- Let $G = H + BK$. The design of matrix $K \in \mathbb{R}^{n \times m}$ such that the closed-loop system is GES relies on:

**Sufficient condition for exponential stability**

[Coron, Bastin, d’Andréa-Novel; 2008]

$$\rho_1(G) = \inf \left\{ \| \Delta G \Delta^{-1} \| ; \Delta \in \mathcal{D}_{n,+} \right\} < 1$$

$\mathcal{D}_{n,+}$: set of diagonal positive definite matrices.

**Proposition [Diange, Bastin, Coron; 2012]**

$\rho_1(G) < 1$ if and only if there exist $\mu > 0$, and a diagonal positive definite matrix $Q \in \mathbb{R}^{n \times n}$ such that the following matrix inequality holds

$$G^T Q \Lambda G < e^{-2\mu} Q \Lambda. \quad (7)$$
Let us consider the Lyapunov function

\[ V(y) = \int_0^1 y(x)^T Q y(x) e^{-2\mu x} \, dx \]  

(8)

Let us note that for all \( t > \frac{1}{\lambda} \) with \( \lambda = \min_{i=1,\ldots,n}\{\lambda_i\} \),

\[ V(y(t, \cdot)) = \sum_{i=1}^n Q_{ii} \int_0^1 \left[ H_i y(t - \frac{x}{\lambda_i}, 1) + B_i u(t - \frac{x}{\lambda_i}) \right]^2 e^{-2\mu x} \, dx \]  

(9)

Computing the right time-derivative of \( V \), we obtain

\[ D^+ V = y^T (\cdot, 0) Q \Lambda y (\cdot, 0) - y^T (\cdot, 1) e^{-2\mu} \Lambda y (\cdot, 1) - 2\mu \int_0^1 y^T (\Lambda e^{-2\mu x} Q) y \, dx \]
Under the sufficient condition for stability $\rho_1(G) < 1$, we get strict Lyapunov condition $+$ ISS property

$$D^+V \leq -2\nu V + \rho \|d\|^2$$  \hspace{1cm} (10)

for suitable values of $\nu$ and $\rho$.

We rewrite

$$D^+V \leq -2\nu(1 - \sigma)V + (-2\nu\sigma V + \rho \|d\|^2), \quad \sigma \in (0, 1)$$  \hspace{1cm} (11)

Therefore, we introduce the increasing sequence of time instants $(t^u_k)$ being defined iteratively by $t^u_0 = 0$, $t^u_1 = \frac{1}{\lambda}$ and for all $k \geq 1$,

$$t^u_{k+1} = \inf \{t \in \mathbb{R}^+ | t > t^u_k \land \|BK(-y(t, 1) + y(t^u_k), 1)\|^2 \geq \frac{2\sigma\nu}{\rho}V(t)\}$$  \hspace{1cm} (12)
Under the sufficient condition for stability $\rho_1(G) < 1$, we get \textbf{strict} Lyapunov condition + ISS property

\[
D^+V \leq -2\nu V + \rho \|d\|^2
\]  

for suitable values of $\nu$ and $\rho$.

We rewrite

\[
D^+V \leq -2\nu (1 - \sigma)V + (-2\nu\sigma V + \rho \|d\|^2), \quad \sigma \in (0, 1)
\]

Therefore, we introduce the increasing sequence of time instants $(t^u_k)$ being defined iteratively by $t^u_0 = 0$, $t^u_1 = \frac{1}{\lambda}$ and for all $k \geq 1$,

\[
t^u_{k+1} = \inf\{t \in \mathbb{R}^+|t > t^u_k \land \|BK(-y(t, 1) + y(t^u_k, 1))\|^2 \geq \frac{2\sigma\nu}{\rho}V(t)\}
\]
We define the event-based controller $\varphi_1$

**Definition**

Let $\varphi_1$ be the operator which maps $y(\cdot, 1)$ to $u$ such that

- $t^u_0 = 0$, $t^u_1 = \frac{1}{\Delta}$ and for all $k \geq 1$,

$$t^u_{k+1} = \inf\{t \in \mathbb{R}^+ | t > t^u_k \wedge \|BK(-y(t, 1) + y(t^u_k, 1))\|^2 \geq \frac{2\sigma \nu}{\rho} V(t) + \varepsilon_1(t)\}$$

$$\varepsilon_1(t) = \varsigma V(\frac{1}{\Delta}) e^{-\eta t} \text{ for all } t \geq \frac{1}{\Delta}, \text{ is a decreasing function which is crucial to prove that there is no Zeno phenomena;}$$

- Note that due to (9), the triggering condition (13) is a function of the output only.

**Theorem (Espitia, Girard, Marchand, Prieur; 2016)**

The system $\mathcal{P}$ with controller $u = \varphi_1(y(t, 1))$ has a unique solution and is globally exponentially stable. Moreover, it holds for all $t \geq \frac{1}{\Delta}$,

$$D^+ V(t) \leq -2\nu(1 - \sigma) V(t)$$
Running example.
Network of conservation laws without ODE coupling
Total output flow of the network

- Continuous time controller $\varphi_0$ (black line)
- ISS-static event-based controller $\varphi_1$ (red line)
- $D^+$-event-based controller $\varphi_2$ (blue dashed-dot line)
- ISS-dynamic event-based controller $\varphi_3$ (green dashed line)
Stabilization of linear hyperbolic systems via event-triggered sampling and quantization (Chp 3).

\[ \frac{\partial}{\partial t} y(t, x) + \Lambda \frac{\partial}{\partial x} y(t, x) = 0 \]
\[ y(t, 0) = H y(t, 1) + B u(t) \]

Contributions on this framework:

- Event-triggered mechanisms \((Lyapunov-based)\) to sample the output \(y(t, 1)\);
- Well-posedness and ISS with respect to measurements errors (related to quantization) \(\rightarrow\) Practical stability in \(H^1\) norm.
Event-based control via Backstepping approach (Chapter 4)
Let us consider the unstable $2 \times 2$ linear hyperbolic system

**Original system**

\[
\begin{align*}
    u_t(t, x) + \lambda_1 u_x(t, x) &= c_1 v(t, x) \\
    v_t(t, x) - \lambda_2 v_x(t, x) &= c_2 u(t, x) \\
    \text{B.C} & \quad u(t, 0) = qv(t, 0) \\
    & \quad v(t, 1) = U(t)
\end{align*}
\]

- Some assumptions on the system parameters (well-posedness issues under event-based control);
- Open loop unstable. Mainly due to coupling terms;
- To stabilize it: **Backstepping method** [Vazquez, Krstic, Coron; 2011];
- Full-state feedback.
Main idea: to transform the original “unstable” system into a stable one.

Target system: linear hyperbolic system of conservation laws.

\[
\begin{align*}
\alpha_t(t,x) + \lambda_1 \alpha_x(t,x) &= 0 \\
\beta_t(t,x) - \lambda_2 \beta_x(t,x) &= 0 \\
\text{B.C.} &\quad \alpha(t,0) = q \beta(t,0) \\
&\quad \beta(t,1) = 0
\end{align*}
\]

- Global exponentially stable;
- Nice stability properties, e.g. finite time convergence to the origin;
- We know how to deal with.
Backstepping Volterra transformation which maps the original system into the target system:

\[
\alpha(t, x) = u(t, x) - \int_0^x K^{uu}(x, \xi)u(t, \xi)d\xi - \int_0^x K^{uv}(x, \xi)v(t, \xi)d\xi
\]

\[
\beta(t, x) = v(t, x) - \int_0^x K^{vu}(x, \xi)u(t, \xi)d\xi - \int_0^x K^{vv}(x, \xi)v(t, \xi)d\xi
\]

Inverse transformation which maps the target system into the original system:

\[
u(t, x) = \alpha(t, x) + \int_0^x L^{\alpha\alpha}(x, \xi)\alpha(t, \xi)d\xi + \int_0^x L^{\alpha\beta}(x, \xi)\beta(t, \xi)d\xi
\]

\[
v(t, x) = \beta(t, x) + \int_0^x L^{\beta\alpha}(x, \xi)\alpha(t, \xi)d\xi + \int_0^x L^{\beta\beta}(x, \xi)\beta(t, \xi)d\xi
\]

- Kernels \(K\) and \(L\) are well-posed.
The backstepping transformation is used to get $U(t)$ under the form ([Vazquez, Krstic, Coron; 2011])

$$U(t) = \int_0^1 K^{vu}(1, \xi)u(t, \xi)d\xi + \int_0^1 K^{vv}(1, \xi)v(t, \xi)d\xi$$

or

$$U(t) = \int_0^1 L^{\alpha}(1, \xi)\alpha(t, \xi)d\xi + \int_0^1 L^{\beta}(1, \xi)\beta(t, \xi)d\xi$$ (16)
Event-based control

Original system

\[
\begin{align*}
    u_t(t, x) + \lambda_1 u_x(t, x) &= c_1 v(t, x) \\
    v_t(t, x) - \lambda_2 v_x(t, x) &= c_2 u(t, x) \\
    \text{B.C} \quad u(t, 0) &= qv(t, 0) \\
    v(t, 1) &= U_d(t)
\end{align*}
\]

Target “perturbed” system

\[
\begin{align*}
    \alpha_t(t, x) + \lambda_1 \alpha_x(t, x) &= 0 \\
    \beta_t(t, x) - \lambda_2 \beta_x(t, x) &= 0 \\
    \text{B.C} \quad \alpha(t, 0) &= q\beta(t, 0) \\
    \beta(t, 1) &= d(t)
\end{align*}
\]

- \( U_d(t) = U(t) + d(t) \);
- \( d \) can be seen as a deviation between the continuous-time controller and the event-based one.

\[
d(t) = \int_0^1 L^{\beta} (1, \xi) \alpha(t_k, \xi) d\xi + \int_0^1 L^{\beta} (1, \xi) \beta(t_k, \xi) d\xi \\
    - \int_0^1 L^{\beta} (1, \xi) \alpha(t, \xi) d\xi - \int_0^1 L^{\beta} (1, \xi) \beta(t, \xi) d\xi
\]

(17)
Event-based controller $\varphi$

**Definition**

We define $\varphi$ the operator from $C^0(\mathbb{R}^+; L^2([0, 1]; \mathbb{R}^2))$ to $C_{pw}(\mathbb{R}^+, \mathbb{R})$ that maps $(\alpha, \beta)^T$ to $U_d$ as follows:

- **Let the increasing sequence of time instants** $(t_k)$ be defined iteratively by $t_0 = 0$, and for all $k \geq 1$,

$$t_{k+1} = \inf\{t \in \mathbb{R}^+ | t > t_k \land \theta Be \mu d^2 \geq \theta \sigma \nu V(t) - m(t)\} \quad (18)$$

where $m$ satisfies the ordinary differential equation,

$$\dot{m}(t) = -\eta m(t) + \left( Be \mu d^2 - \sigma \nu V(t) - \kappa_1 \alpha^2(t,1) - \kappa_2 \beta^2(t,0) \right) \quad (19)$$

- **Let the control function** be defined by:

$$U_d(t) = \int_{0}^{1} L^\beta \alpha(1,\xi) \alpha(t_k,\xi) d\xi + \int_{0}^{1} L^\beta \beta(1,\xi) \beta(t_k,\xi) d\xi \quad (20)$$

for all $t \in [t_k, t_{k+1})$. 


On the existence of the minimal dwell-time

- Inspired by [Tabuada; 2007] and [Girard; 2015].

Events are triggered to guarantee, for all $t > 0$,

$$\theta Be^\mu d^2(t) \leq \theta \sigma \nu V(t) - m(t)$$

- $\psi$ is continuous on $[t_k, t_{k+1}]$;
- $m(t) \leq 0$.

$$\psi = \frac{\theta Be^\mu d^2 + \frac{1}{2} m}{\theta \sigma \nu V - \frac{1}{2} m}$$
**Event-based stabilization**

**EBC-Backstepping approach**

\[ \psi(t_{k+1}) = 1 \]

\[ \psi(t) \leq \Psi(t) \quad \forall t \in [t'_k, t_{k+1}] \]

\[ \dot{\psi} = \frac{2\theta B e^{\mu} d\dot{d} + \frac{1}{2} \dot{m}}{\theta \sigma v V - \frac{1}{2} m} - \frac{(\theta \sigma \dot{V} - \frac{1}{2} \dot{m})}{\theta \sigma v V - \frac{1}{2} m} \psi \]

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Event-based stabilization

EBC-Backstepping approach

Conclusion

\[
\dot{\psi} = \frac{2\theta Be^\mu d\dot{d} + \frac{1}{2}\dot{m}}{\theta \sigma \nu \dot{V} - \frac{1}{2}m} - \frac{(\theta \sigma \nu \dot{V} - \frac{1}{2}\dot{m})}{\theta \sigma \nu \dot{V} - \frac{1}{2}m} \psi
\]

\[\psi(t) \leq \Psi(t) \quad \forall t \in [t'_k, t_{k+1}]\]

\[ \dot{\psi} \leq a_0 + a_1\psi + a_2\psi^2 \]

\[ a_0 = \frac{Be^\mu \varepsilon_3}{\sigma v} + \eta + \varepsilon_2 + 1 \]
\[ a_1 = -\sigma v + \mu \max\{\lambda_1, \lambda_2\} + \eta + \varepsilon_2 + 1 + \frac{1}{2\theta} \]
\[ a_2 = -\sigma v + \frac{1}{2\theta} \]

Then, by the **Comparison principle**, \( \psi(t) \leq \Psi(t) \) where \( \Psi \) is the solution of

\[ \dot{\Psi} = a_0 + a_1\Psi + a_2\Psi^2 \]

It follows that **the time needed by** \( \psi \) (or \( \Psi \)) **to go from** \( \psi(t_k') = 0 \) (or \( \Psi(t_k') = 0 \)) **to** \( \psi(t_{k+1}) = 1 \) (or \( \Psi(t_{k+1}) = 1 \)) **is at least**

\[ \tau = \int_0^1 \frac{1}{a_0 + a_1s + a_2s^2}ds \]

Thus,

\[ t_{k+1} - t_k \geq t_{k+1} - t_k' \geq \tau \]

**\( \tau \)** **a lower bound of the inter-execution times or minimal dwell-time.**
Numerical simulation

- Transport velocities: $\lambda_1 = 1$, $\lambda_2 = \sqrt{2}$,
- Coupling terms: $c_1 = 1.5$, $c_2 = 2$ and $q = 1/4$.
- Initial conditions: $u^0(x) = q v^0(x)$ with $v^0(x) = 10(1 - x)$ for all $x \in [0, 1]$.

Trajectories involved in the triggering condition.
Time-evolution of

- Continuous-time control $U$ (black dashed line)
- Event-based control $U_d$ (blue line with red circle marker)
Second components $v(t, x)$ of the closed-loop system

under $U(t)$  

under $U_d(t)$
Conclusion and Perspectives

- Modeling of communication networks: coupled PDE-ODEs.
- Input-to-state stability of coupled PDE-ODEs and boundary control synthesis.
- Extension of event-based controls developed for finite-dimensional systems to linear hyperbolic systems by means of Lyapunov techniques;
- Backstepping approach (full-state);
  - Existence of a minimal dwell-time;
- Global exponential stability and well-posedness of the system under E.B.C.
Short term perspectives

- To find a “period” using *looped functionals* in order to impose a dwell-time (In progress).

**Periodic event-based control.**

\[
\begin{align*}
\frac{\partial}{\partial t} y(t, x) + \Lambda \frac{\partial}{\partial x} y(t, x) &= 0 \\
y(t, 0) &= H y(t, 1) + B y u(t)
\end{align*}
\]

\[
\dot{\eta}(t) = A \eta(t) + B_n y(t_k, 1) \\
u(t) = K \eta(t_j)
\]

- \( \forall t \in [\tau_j, \tau_{j+1}) \) with \( T = \tau_{j+1} - \tau_j \), the period;

Long term perspectives

- Event-triggering conditions depending on observed states (for the Backstepping approach);

- Event-based boundary control of *parabolic equations* for the modeling of networks.
Thank you for your attention!
Optimization issues and control constraints

minimize $\frac{\gamma e^{2\mu}}{2\nu}$

subject to $M_c \leq 0$; $\|K_{zi}\| \leq \frac{p\delta^w_i}{\beta_z}$; $\|K_{yi}\| \leq \frac{(1 - p)\delta^w_i}{\beta_y}$; $\|L_{zi}\| \leq \frac{\delta^u_{i,j}}{\beta_z}$
Change of variables:

- \( X = P^{-1} \);
- \( Q_3 = (Q\Lambda)^{-1} \);
- \( Y_{Kz} = K_z X \), \( Y_{Lz} = L_z X \), \( Y_{Ky} = Ky Q_3 \) and \( Y_{Ly} = Ly Q_3 \)

\( \tilde{M}_c \) is given by

\[
\begin{pmatrix}
    X A^T + A X + 2\mu \Lambda X + Y_{Kz}^T B_w^T + B_w Y_{Kz} & G_y Q_3 + B_w Y_{Ky} & D & X G_{z}^T + Y_{Ly}^T B_u^T \\
    \ast & G_y Q_3 + B_w Y_{Ky} & \ast & \ast \\
    \ast & \ast & -\gamma I & \ast \\
    \ast & \ast & \ast & Y_{Ly}^T B_u^T \\
    \ast & \ast & \ast & -Q_3
\end{pmatrix}
\]
Combined with a line search on $\mu$,

\[
\begin{align*}
\text{minimize} & \quad \frac{\gamma}{2 \nu} e^{2\mu} \\
\text{subject to} & \quad \tilde{M}_c \leq 0; \\
& \quad \begin{pmatrix} \eta H e(X) - \eta^2 I & Y_{K_z} \\ * & \left(\frac{p \delta^w_i}{\beta_z}\right)^2 I \end{pmatrix} \geq 0; \\
& \quad \begin{pmatrix} \eta H e(X) - \eta^2 I & Y_{L_z} \\ * & \left(\frac{p \delta^u_{i,j}}{\beta_z}\right)^2 I \end{pmatrix} \geq 0 \\
& \quad \begin{pmatrix} \eta H e(Q_3) - \eta^2 I & Y_{K_y} \\ * & \left(\frac{(1-p) \delta^w_i}{\beta_y}\right)^2 I \end{pmatrix} \geq 0;
\end{align*}
\]
This approach uses the static triggering condition previously introduced.

**ISS static event-based stabilization**

Events are triggered so that $\|d\|^2 - \kappa V - \varepsilon$ is always less than 0.

We impose that the *weighted averaged value* of $\|d\|^2 - \kappa V - \varepsilon$ is less than 0.

**Internal dynamic:**

$$m(t) = e^{-\eta t} \int_{1/\lambda}^{t} e^{\eta s} (-\kappa V(s) - \varepsilon(s) + \|d(s)\|^2) ds \quad \forall t \geq \frac{1}{\lambda}$$

Consider the following Lyapunov function candidate $W$,

$$W(y, m, \varepsilon) = V(y) + \frac{\rho}{\eta - 2\nu(1-\sigma)} \varepsilon - \rho m \quad (21)$$
Computing the right time-derivative of $W$

$$D^+ W = D^+ V - \eta \frac{\rho}{\eta - 2\nu(1 - \sigma)} \varepsilon - \rho ( - \eta m - \kappa V - \varepsilon + \|d\|^2)$$

\[ \Rightarrow D^+ W(t) \leq -2\nu (1 - \sigma) W(t) + \rho (-2\nu (1 - \sigma) + \eta) m(t) \] (22)
We define the event-based controller $\varphi_3$

**Definition**

Let $\varphi_3$ be the operator which maps $z$ to $u$ such that

- $t^u_0 = 0$, $t^u_1 = \frac{1}{\lambda}$ and for all $k \geq 1$,
  \[
  t^u_k+1 = \inf\{t \in \mathbb{R}^+ | t > t^u_k \wedge m(t) \geq 0\} \tag{23}
  \]

- $u(t) = Ky(t^u_k, 1) \quad \forall t \in [t^u_k, t^u_{k+1}), \ k \geq 0 \tag{24}$

**Theorem (Espitia, Girard, Marchand, Prieur; NOLCOS 2016)**

The system $\mathcal{P}$ with controller $u = \varphi_3(y(t, 1))$ has a unique solution and is globally exponentially stable.
Vehicle traffic flow on a roundabout

\[ \frac{\partial}{\partial t} y(t, x) + \Lambda \frac{\partial}{\partial x} y(t, x) = 0 \]

\[ y(t, 0) = (H + BK)y(t, 1) \]

with

- \( \Lambda = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{2} \end{pmatrix} \),
- \( H = \begin{pmatrix} 0 & 0.7 \\ 0.9 & 0 \end{pmatrix} \),
- \( B = I_2 \),
- \( K = \begin{pmatrix} 0 & 0.3 \\ -0.9 & 0 \end{pmatrix} \).
\( NT = 8000 \) with \( \Delta t = 1 \times 10^{-3} \).

- Mean value of triggering times: 158.3 events;
- Mean value \textbf{inter-execution times}: 0.0432.

\( \varphi_1 \)

- Mean value of triggering times: 109.1 events;
- Mean value \textbf{inter-execution times}: 0.0640.

\( \varphi_3 \)
Theorem

Let $\sigma \in (0, 1)$, $\mu > 0$, $v = \mu \min\{\lambda_1, \lambda_2\}$, $A = e^\mu$, $B = e^\mu q^2 + 1$, $\varepsilon_1$. Let $\eta \geq v(1 - \sigma)$ and $0 < \theta \leq \min\{\frac{1}{2\sigma v}, \frac{1}{2Be^\mu \varepsilon_1}\}$, $\kappa_1$ and $\kappa_2$ such that

$$\max\{2\theta Be^\mu \varepsilon_1, 2\theta \sigma v\} \leq \kappa_1 \leq 1 \quad (25)$$

$$2\theta \sigma v \leq \kappa_2 \leq 1 \quad (26)$$

holds. Then the system (17)-(17) with event-based controller $U_d = \varphi$ has a unique solution and is globally exponentially stable.

$$W(\alpha, \beta, m) = \int_0^1 (A\alpha^2(x)\frac{e^{-\mu x}}{\lambda_1} + B\beta^2(x)\frac{e^{\mu x}}{\lambda_2})dx - m \quad (27)$$

$$\dot{W} \leq -v(1 - \sigma)W$$
Periodic event-based control.

\[
\begin{aligned}
\frac{\partial}{\partial t} y(t, x) + \Lambda \frac{\partial}{\partial x} y(t, x) &= 0 \\
y(t, 0) &= H y(t, 1) + B_y u(t)
\end{aligned}
\]

\[u(t) = K \eta(t_{\tau_j})\]

\[\eta(t) = A \eta(t) + B_{\eta} y(t_{\tau_j}, 1)\]

\[\forall t \in [\tau_j, \tau_{j+1}) \text{ with } T = \tau_{j+1} - \tau_j, \text{ the period;}
\]

Can we obtain a sufficient condition for stability in terms of the period \(T\)?

**Looped functional**

\[
W(\eta, y, t - \tau_j) = V(\eta, y) + V(\eta, t - \tau_j)
\]

\[
V(\eta, t - \tau_j) = (\tau_{j+1} - t)(\eta(t) - \eta(\tau_j))^T S_1(\eta(t) - \eta(\tau_j)) + (\tau_{j+1} - t) \int_{\tau_j}^t \dot{\eta}(\theta) R \dot{\eta}(\theta) d\theta
\]


Event-based control of linear hyperbolic systems of conservation laws. 
*Automatica*, 70:275–287.

Dynamic triggering mechanisms for event-triggered control. 

Gas flow in fan-shaped networks: Classical solutions and feedback stabilization. 

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Event-triggered real-time scheduling of stabilizing control tasks. 