In delay there lies no plenty

Time-delay systems are also called systems with after effect or dead-time, hereditary systems, equations with deviating argument or differential-difference equations. They belong to the class of functional differential equations which are infinite dimensional, as opposed to ordinary differential equations. In spite of their complexity, they may often appear as simple infinite-dimensional models in the playground of partial differential equations. After the presentation of some motivating examples, the talk will try to show main differences arising from the presence of deviating time-arguments in the dynamics, seen from different points of view: state, solutions, stability, identification...

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What's to come is still unsure:
In delay there lies no plenty;
Then come kiss me, sweet and twenty;
Youth's a stuff will not endure.

† Shakespeare, W. Twelfth Night, Or what you will, 2(3), 1699.
Delays : Classical examples

Strejć/Broïda – like models for industrial control
- frequent in process engineering
- simple and generic approximation
- PID controller ? ... poor results if $T > \tau$
  ➔ Smith predictor or « GPID » ...

![Diagram of temperature changes and mathematical model](image)

Delays : Another classical example

... a great standard of control classes (Feedback)

... and of systems with transport phenomena

$T \approx 20.10^{-3} \text{sec}$, $\tau \approx 1 \text{ sec.}$

PID ok
Ex Thomas Erneux
(a smarter system with transport phenomenon)
Much a do about delay?

- applied problem engineering (NCS, telecom, finance, biology, populations, nuclear...)
- still open in many cases closed-loop, variable delays, unknown delays, identification...
- the « simplest » infinite dimension problem functional equations, particular case of PDEs
- surprising properties damaging/improving by adding delays, data-sampling model...
Contents

Illustrative examples
- 1st example (remote ctrl.) → basic notions (stability, state, inf. dim.) goodies: chaos
- 2nd example : variable delay → counter-example
- 3rd example : sampling → delay for modelling ZoH
- 4th example : Networked Control System (master-slave)

Cauchy’s problem for TDS
- Notion of solution
- Lipschitz-type condition
- And if the delay can vanish...

Stability and Lyapunov
- the LTI case
- 1st Lyapunov (small states)
- small delays
- 2nd Lyapunov

Some words about identification

A simple example

expected angle $\dot{x} = 0$

$\text{error } e = 0 - x$

voltage $u$

drive

measured angle $x$

speed $\dot{x}(t) = ku(t)$, $\frac{X(s)}{U(s)} = \frac{k}{s}$

feedback:

$\dot{x}(t) = ku(t) = -kx(t)$

$\dot{x}(t) + kx(t) = 0$

ODE
Simple example (Cont’d)

**Embedded drive (satellite)**

expected angle $x = 0$

remote feedback:
$u(t) = k\varepsilon(t - h/2) = -kx(t - h)$

transmitted control $\varepsilon(t-h/2)$

transmitted angle $x(t-h/2)$

Paris-LA
$h/2 \approx 50 \times 10^{-3}$ sec (15 $10^2$m)

$h/2 = 1.28$ sec (0.4 $10^2$m)

$h/2 \approx 260$ à 1260 sec...

$\dot{x}(t) + kx(t - h) = 0$

Simple example (cont’d)

$\dot{x}(t) + x(t - h) = 0$

(case $h = 1$, $k = 1$)

$\dot{x}(t) = -x(t - 1)$

C.I. $t = 0 : x(t) = 1$ ??

$t \in [-1, 0] : x(t) = 1$ (C.I.)

$t \in [0, 1] : x(t) = 1 - t$,

$t \in [1, 2] : x(t) = \frac{1}{2} - t + \frac{t^2}{2}$,

etc.

and w.r.t. $h$?
Simple example (cont’d)

\[ \dot{x}(t) + x(t - h) = 0 \]

w.r.t. \( h \)?

\[ \dot{x}(t) + x(t - 1) = 0 \]

to be compared with \((h=0)\):

\[ \dot{x}(t) + x(t) = 0 \]

Simple example (cont’d)

\[ \dot{x}(t) + x(t - 1.6) = 0 \]

\( h = 1.6 \)

\( t \in [-1, 0] : x(t) = 1 \) (same I.C.)
a delay can have a stabilizing effect as well

here, derivative effect: $y(t-h) \approx y(t) - h \dot{y}(t)$

Simple example (cont’d)

$\dot{x}(t) + x(t-h) = 0 \quad h = \frac{\pi}{2}$

(Shimanov’s notation, 1960)

$\dot{x}(t) = f(x(t), t, u(t)), \quad t \geq t_0,$

$x(t) = x(t + \theta), \quad -h \leq \theta \leq 0,$

$u(t) = u(t + \theta), \quad -h \leq \theta \leq 0,$

$x(\theta) = \varphi(\theta), \quad t_0 - h \leq \theta \leq t_0,$

$\Rightarrow$ «state» notion?

a variable $\varphi(t)$ generating a unique solution starting at instant $t$

function $x_t = \text{state at time } t$

vector $x(t) = x_t(0)$ solution at $t$

function $x_t$ infinite dim. syst.
A note about chaos

- ODEs: no chaos for differential order < 3
- FDEs: possible for n=2...or less?

\[ \dot{x}(t) = -\sin(x(t) - \tau) \]
\[ \dot{x}_0 = \sin(t_N), \]
\[ \dot{x}_i = N(x_{i-1} - x_i)/\tau, \]
where \( 1 \leq i \leq N \rightarrow \infty. \)

Simple example (cont’d)

\[ \dot{x}(t) + x(t - h) = 0 \quad h = 1 \]

\[ s + e^{-s} = 0 \]

\[ s = \alpha + j\beta \]
\[ \alpha + e^{\alpha} \cos \beta = 0 \]
\[ j(\beta - e^{\alpha} \sin \beta) = 0 \]

inf. poles \quad infinite dim. syst.
Simple example (cont’d)

\[ \dot{x}(t) + x(t - h) = 0 \]

\[ h = 1 \]

\[ s + e^{-s} = 0 \]

\[ s e^{-s} + e^{-s} = 0 \]

stability criterion?

retarded \rightarrow \text{Hurwitz ok}

frequency behavior?
(Bode, open loop)

\[ \varphi = -\pi/2 - jh\omega \]

phase \rightarrow \infty

phase \rightarrow -\infty \rightarrow \text{infinite dim. …
Simple example (the end!)

\[ \dot{x}(t) + x(t - h) = 0 \]

**to sum up...**

- delay ⇒ strong influence on stability
- fonctionnal state
- infinite nb of poles (Hurwitz OK, Routh no)
- strong phase displacement (\( \rightarrow -\infty \))

**and, up to now, it was not that complicated**

- constant delay
- linear, scalar system « 1st order »

**is it the same for variable delays \( h(t) \)?**

- a counter-example...
(counter-)example with variable delay

\[ \dot{x}(t) = -ax(t) - bx(t - h(t)) \]  
\[ h(t) = t - kT \quad \text{for} \quad kT < t \leq (k+1)T \]

is asymptotically stable iff (yellow zone):

\[ (1 + \frac{b}{a} e^{-aT} - \frac{b}{a}) < 1 \quad \text{if} \quad a \neq 0 \]
\[ |1 - bT| < 1 \quad \text{if} \quad a = 0 \]

and, for \( h = \text{cst} \in [0,1] \) iff (pink zone)

OK, but does such a delay \( h(t) \) happen? another example...

Sampled systems: an interesting idea…
Mikheev et al. 86, Sobolev et al. 89, Astërm et al. 92

\[ u(t) = u_d(t_k) = u_d(t - [t - t_k]) = u(t - h(t)) \]
...with application to aperiodic sampling

\[ u(t) = g(x(nT(t))) \]

Another statement of the same problem...

- Influence of the maximum sampling period \( h_m \)
- Application of Fridman’s criterion \( \frac{dh}{dt} \leq 1 \)
4th example: Networked Control Systems

Seuret et al. ACC 2006, Jiang et al. CDC’09, Kruszewski et al. IEEE CST 2011...

\[
i(t) = A_i(t) + B_i(t - h_1(t)), \quad y_i(t) = C_i(t).
\]

\[ h_i + \tau_i = \delta_i \rightarrow \text{homogeneous delay formulation} \]

A week of RTT...

Mean = 82 ms
Maxi = 857 ms
Mini = 1 ms
15 ms

Global delay

State variables

Adaptation of control/observation gains $K$, $L$ to the available QdS

Note the predictor effect (compensates the communication delay)

Mixing tank system (with total recycle)

- Control = delay = $k$ (motor speed)$^{-1}$
- Non flat system

$T(u(t))\dot y(t) = y(t) - h(u(t)) - y(t)$.

Goodies

An arrangement of ideal zones with shifting boundaries as a way to model mixing processes in unsteady-stirring conditions in agitated vessels

Goodies (2)

Design of a Pressure Control System With Dead Band and Time Delay

Jan Anthonis, Alexandre Seuret, Jean-Pierre Richard, and Herman Ramon
IEEE TRANSACTIONS ON CONTROL SYSTEMS TECHNOLOGY, VOL. 15, NO. 6, NOVEMBER 2007

→ deadzone (dry friction on angle $\varphi_1$)
+ delay (measured pressure $y$)
+ sign functions (control)

Pressure $y$ ruled by the angle $\varphi_1$

$$\sqrt{y} = \frac{c\varphi_1(t-h) + \beta}{\varphi_1(t-h) + \gamma}, \quad 0 \leq h_{\text{min}} \leq h \leq h_{\text{max}}$$

Delays: various other examples

Applied Delay Differential Equations

250 references taken from

- life sciences
- physics
- technology
- chemistry
- economics
OK, that's enough examples,
let's go to general questions

## Contents

<table>
<thead>
<tr>
<th>Illustrative examples</th>
<th>Distinctive features of TDS?</th>
</tr>
</thead>
<tbody>
<tr>
<td>1&lt;sup&gt;st&lt;/sup&gt; example (remote ctrl.)</td>
<td>basic notions (stability, state, inf. dim.)</td>
</tr>
<tr>
<td>2&lt;sup&gt;nd&lt;/sup&gt; example : variable delay</td>
<td>counter-example</td>
</tr>
<tr>
<td>3&lt;sup&gt;rd&lt;/sup&gt; example : sampling</td>
<td>delay for modelling</td>
</tr>
<tr>
<td>4&lt;sup&gt;th&lt;/sup&gt; example : Networked Control System (master-slave)</td>
<td></td>
</tr>
</tbody>
</table>

### Cauchy's problem for TDS
- Notion of solution
- Lipschitz-type condition
- And if the delay can vanish...

### Stability and Lyapunov
- the LTI case
- 1<sup>st</sup> Lyapunov (small states)
- small delays
- 2<sup>nd</sup> Lyapunov

Some words about identification

(just a little bit)
Cauchy’s problem
(existence and unicity of solution for a TDS)

- Notion of solution
- Lipschitz-type condition
- And if the delay can vanish...

2.3 Notion of solution

System (S) : \( \dot{x}(t) = f(t, x(t), x(t - \tau(t))) \)

with \( x(t) \in \mathbb{R}^n \), and \( 0 \leq \tau(t) \leq \tau \)

Let \( \varphi : [-\tau, 0] \rightarrow \mathbb{R}^n \) be an arbitrary map.

Definition: A map \( x(t) : [t_0 - \tau, t_0 + b) \rightarrow \mathbb{R}^n \) s.t.
1) \( x(t_0 + s) = \varphi(s) \), for all \( s \) in \([-\tau, 0]\);
2) \( x \) is continuous over \([t_0, t_0 + b] \);
3) \( x \) satisfies (S) over \([t_0, t_0 + b) \) (\( \dot{x} \) right-hand, Dini)

is called a solution of (S) with initial value \( \varphi \) at \( t_0 \).

If only one map satisfies these 3 points, then the solution is unique.

Remark: There is a weaker notion of solution, where
2) \( x \) is absolutely continuous function over \([t_0, t_0 + b) \)
3) \( x \) satisfies (S) almost everywhere on \([t_0, t_0 + b) \)
2.4 Existence and uniqueness of solutions

For system (S) with $0 < \delta \leq \tau(t) \leq \tau_m$:

\[ \dot{x}(t) = f(t, x(t), x(t - \tau(t))). \]

Consequence of the step method:
Given a continuous map $\varphi \in C$, if the ODE

\[ \dot{x}(t) = f(\varphi(t, x(t)) \equiv f(t, x(t), \varphi(t - \tau(t))) \]

has a (unique) solution, then there exists a (unique) solution of (S) with initial condition $\varphi$.

From there, using classical Cauchy-Lipschitz conditions:

\[ \text{Conditions of existence and uniqueness (I)}: \]

If $f$ is a continuous map and satisfies a local Lipschitz condition in $x$,

\[ \|f(t, x_2, y) - f(t, x_1, y)\| \leq K \|x_2 - x_1\|, \]

then for any initial condition $\varphi \in C$, (S) has a unique solution, depending continuously on $f$ and $\varphi$.

If the delay can become zero, $0 \leq \tau(t) \leq \tau_m$ the step method does not apply anymore

⇒ need of a general framework: FDEs [Myshkis 49]

(RFDE): \[ \dot{x}(t) = F_R(t, x_t) \] (retarded type)

Conditions of existence and uniqueness (II):

If $F_R$ is a continuous map with local-Lipschitz cond.
in its second (functional) argument, i.e.

\[ \|F_R(t, \varphi_2) - F_R(t, \varphi_1)\| \leq K \|\varphi_2 - \varphi_1\|, \]

then for any initial condition $\varphi \in C([-\tau, 0], \mathbb{R}^n)$, (RFDE) has a unique solution, depending continuously on $F_R$ and $\varphi$.

This condition is also necessary for previous system (S) (discrete, bounded, nonzero delay)
Remark:

Even if unicity holds, different solutions may coincide after a finite time. For instance:

\[ x(t) = -x(t - \tau)[1 - x(t)] , \]

\[ x(t, \varphi) = 1 \quad (\forall t \geq 0) \]

for any \( \varphi \in C([-\tau, 0], \mathbb{R}) \) such that \( \varphi(0) = 1 \).

(non-unicity of the trajectory reversion)

Contents

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  - 1st example (remote ctrl.) \( \rightarrow \) basic notions (stability, state, inf. dim.)
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  - 3rd example : sampling \( \rightarrow \) delay for modelling
  - 4th example : Networked Control System (master-slave)

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Some words about identification
Stability

- the LTI case
- 1\textsuperscript{st} Lyapunov (small states)
- small delays
- 2\textsuperscript{nd} Lyapunov

Stability: the LTI case

Theorem:

A linear time-invariant system (thus, with a constant delay):

\[ \dot{x}(t) = A_0 x(t) + A_1 x(t - h), \]

is globally asymptotically stable if all its characteristic roots:

\[ \det(sI - A_0 - e^{-hs}A_1) = ... 0 \]

are in the strict left half plane.

Exemple 1. Considérons l’équation

\[ \dot{x}(t) = -x(t - 1) \]

Son équation caractéristique est \( s + e^{-s} = 0 \), dont les solutions \( s = a \pm j\beta \) sont en nombre infini.

Le système n’est donc pas dégénéré. Ici, \( s = -0.318 \pm 1.337j \) est une estimation de la paire de racines de plus grande partie réelle : il y a donc stabilité asymptotique\(^2\). Par contre, le cas suivant est dégénéré et instable :

\[ \dot{x}(t) = \begin{pmatrix} 0 & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \\ 0 & -\frac{1}{2} & 0 \end{pmatrix} x(t) + \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} x(t - h) \]

\[ \det(sI - A_0 - e^{-hs}A_1) = ... s(s^2 - 1). \]
Stability: 1st method of Lyapunov

Approximation of the small deviations

\[
\dot{z}(t) = \sum_{i=0}^{k} A_i x(t-h_i) + q(t, x_t)
\]

\[
q(t, x_t) = q(t, x(t), x(t-\tau_1(t)), \ldots, x(t-\tau_k(t)));
\]

\[
h_0 = 0, \ h_i = \text{constantes}, \ \tau_j(t) \in [0, \tau_i] \text{ continues},
\]

\[
\|u_i\| \leq \varepsilon \Rightarrow \|q(t, u_0, \ldots, u_k)\| \leq \beta_\varepsilon (\|u_0\| + \ldots + \|u_k\|),
\]

with \(\beta_\varepsilon \) constant for \(\varepsilon \) given, \(\beta_\varepsilon \) uniformly decreasing towards 0 when \(\varepsilon \rightarrow 0\). The approximation first order is defined by:

\[
\dot{z}(t) = \sum_{i=0}^{k} A_i z(t-h_i).
\]

If the linearized system (5.9) is asymptotically stable, then \(z = 0\) is asymptotically stable for (5.8).

If (5.9) has at least one characteristic root with positive real part, then \(z = 0\) is unstable for (5.8).

Stability: LTI with small delays

Approximation of small delays

\[
\dot{z}(t) = \sum_{i=0}^{k} A_i z(t-h_i). \quad (5.9)
\]

\[
\dot{\hat{z}}(t) = \sum_{i=0}^{k} A_i \hat{z}(t-h_i) + q(t, x_t). \quad (5.8)
\]

The previous result (small variation approximation) can be usefully completed with a small delay approximation, which is a qualitative result obtained from the continuity of the characteristic roots of (5.9) w.r.t. the delays \(h_i\).

Theorem:
If \(A\) is Hurwitz (resp., unstable), then for sufficiently small values of the delays \(h_i\), \(z = 0\) is asymptotically stable (resp., unstable) for (5.9) and, thus, for (5.8).

If \(A\) has one zero eigenvalue and all the other ones have negative real parts, then for sufficiently small values of the delays \(h_i\), \(z = 0\) is stable (resp., unstable) for (5.9) and, thus, for (5.8).
Stability: LTI with small, single delay

\[ \frac{dz(t)}{dt} = A_0 z(t) + A_1 z(t-h), \]  

(5.10)

qui, pour un retard nul, devient :

\[ \frac{dz(t)}{dt} = (A_0 + A_1) z(t). \]  

(5.11)

**Sufficient condition**

**Theorem 6.** [40] Si le système à retard nul (5.11) est asymptotiquement stable et si \( P \) est la matrice solution de l'équation de Liapounov (où \( Q \) est une matrice réelle définie positive [117]) :

\[ (A_0 + A_1)^T P + P (A_0 + A_1) = -Q^T Q, \]  

(5.12)

alors (5.10) est asymptotiquement stable pour tout retard \( h \in [0, h_{\text{max}}] \):

\[ h_{\text{max}} = \frac{1}{2} \left[ \lambda_{\text{max}}(B^T B) \right]^{-\frac{1}{2}}, \text{ avec } B = Q^{-T} A_1^T P (A_0 + A_1) Q^{-1}. \]  

(5.13)

Another result by V.B Kolmanovskii & A.D. Myshkis (1999)

\[ \dot{x}(t) = Ax(t-h) \]  

(also in nonlinear \( \dot{x}(t) = f(t,x(t-h)) \)

« dissipative systems »

Lyapunov for TDS

\[ V(x(t)) = \|x(t)\| \text{ (some norm)} \]

\[ \|A\| \text{ = associated matrix norm,} \]

\[ \gamma(A) \text{ = logarithmic norm ("measure").} \]

\[ \gamma(A) < -h\|A\|^2 \implies \text{expon. stable, } e^{\omega t} \]

\[ \omega : \text{solution of } \omega = -\gamma(A) - h\|A\|^2 e^{2\omega h} \]

\[ \omega : \text{solution of } \omega = -\gamma(A) - h\|A\|^2 e^{2\omega h} \]
Lyapunov’s direct method for TDS

**ODE:**
\[ \dot{x}(t) = -ax(t) \quad \Rightarrow \quad V(x(t)) = x^2(t) > 0 \]
\[ \dot{V}(x(t)) = -2ax^2(t) < 0 \ldots \text{ etc.} \]

**FDE:**
\[ \dot{x}(t) = -ax(t) - bx(t-h) \]
\[ V(x(t)) = x^2(t) \quad (\text{« usual » quadratic}) \]
\[ \dot{V}(x(t)) = -2 \left[ ax^2(t) + bx(t)x(t-h) \right] \leq ... ? \]

need of delay-dedicated methods:
1) Lyapunov-Razumikhin functions (not here)
2) Lyapunov-Krasovskii functionals

an illustration of the Lyapunov-Krasovskii approach

\[ \dot{x}(t) = -ax(t) - bx(t-h) \]
\[ V(x_t) = x^2(t) + |b| \int_{-h}^{0} x^2(t+s)ds \quad \text{(quad + integral)} \]
\[ \dot{V}(x_t) = -2x(t)[ax(t) + bx(t-h)] + |b|[x^2(t) - x^2(t-h)] \leq -2(a-|b|)x^2(t) \quad \ldots \quad \dot{V}(x_t) < 0 \text{ if } |b| < a \]

NB: LK-functionals were used in the above NCS/sampled data proofs (under a much more general form)
A bit more general LKF…

\[ (S) \quad \dot{x}(t) = Ax(t) + Bx(t - \tau), \text{ avec } x(t) \in \mathbb{R}^n. \]

\[ \text{Fonctionnelle :} \quad \mathcal{V}(\varphi) = \varphi^T(0)P\varphi(0) + \int_{-\tau}^{0} \varphi^T(s)S\varphi(s) \, ds \]
avec \( P, S > 0. \)
\[ \Rightarrow \quad \dot{\mathcal{V}}(x) = y^T(t)Qy(t) \]
avec \( Q = \begin{bmatrix} A^T P + PA + S & PB \\ B^T P & -S \end{bmatrix} \) et \( y(t) = \begin{bmatrix} x(t) \\ x(t - \tau) \end{bmatrix} \)
\[ \Rightarrow \text{Stabilité asymptotique i.d.r. si } Q < 0 \text{ (LMI)} \]

and a way to delay-dependent stability…

Hyp. : \((S)\) asymp. stable pour \( \tau = 0 \Rightarrow A = A + B \text{ Hurwitz} \)

\[ \text{Problème} \]
Chercher une borne \( \tau^* \) t.q. stabl. asympt. \( \forall \tau \leq \tau^* \).

\[ \text{Idée} \]
Transformation du modèle à l’aide de la formule de Leibniz-Newton :
\[ x(t) - x(t - \tau) = \int_{-\tau}^{0} \dot{x}(t + s) \, ds \]
\[ (S) \Rightarrow \]
\[ \dot{x}(t) = (A + B)x(t) - B \int_{t-\tau}^{t} (Ax(s) + Bx(s - \tau)) \, ds \]
+ many more general LKFs

see the textbook Mathématiques pour l’Ingénieur

pdf also available at http://hal.inria.fr/hal-00519555_v1/

\[
\dot{x}(t) = \sum_{i=1}^{m} A_i x(t - h_i), \quad (6.47)
\]

\[
A = \sum_{i=1}^{m} A_i, \quad A_{ij} = A_i A_j, \quad h_{ij} = h_i + h_j, \quad h = \sum_{i=1}^{m} h_i. \quad (6.48)
\]

**Theorem 6.5.8.** Le système (6.47) est asymptotiquement stable si, pour deux matrices symétriques et définies positives \( R, Q \), il existe une matrice définie positive \( P \) solution de l’équation de Riccati :

\[
A^T P + PA + m R h + \sum_{j=1}^{m} h_i A_{ij} R^{-1} A_{ij}^T P = -Q. \quad (6.49)
\]

**Démonstration** : on choisit la fonctionnelle \( V = V_1 + V_2 \). \( V_1 = x^T(t) P x(t) \). \( V_2 = \sum_{j=1}^{m} \int_{t-h_j}^{t} x^T(\tau) R x(\tau) d\tau \), conduisant à \( \dot{V} = -x^T(t) Q x(t) - \sum_{j=1}^{m} \int_{t-h_j}^{t} [R x(\theta) + A_{ij}^T P x(t) R^{-1} R x(\theta) + A_{ij}^T P x(t)]^2 d\theta. \)

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… Few last words about identification
Some words on our current research in identification/estimation

Non-A : Non-Asymptotic estimation for on-line systems
http://www.inria.fr/equipes/non-a

a foreword on identifiability
Adaptive identification of linear time-delay systems

Y. Orlov, L. Belkoura, J.P. Richard, and M. Dambrine

\[ \dot{x}(t) = \sum_{i=0}^{m} A_i x(t - \tau_i) + B_i u(t - \tau_i), \]
\[ y(t) = C(\lambda) x(t), \]
\[ r(t) = \lambda(t) \]

over the ring \( \mathbb{R}[\lambda] \) of polynomials in a vector variable \( \lambda = (\lambda_1, \ldots, \lambda_k) \).

In matrix terms system (1) is weakly controllable if for some \( z \in \mathbb{C}^k \),

\[ \text{rank} \left[ B(z) \mid A(z)B(z) \mid \ldots \mid A^{n-1}(z)B(z) \right] = n \]

Definition 1 System (1) is said to be identifiable if there exists a control input \( u(t) \) such that the identity \( x(t) = \hat{x}(t) \) results in

\[ r = \hat{r}, \tau_i = \tau_i, A_i = \hat{A}_i, B_i = \hat{B}_i \] for \( i = 0, \ldots, r, \)

regardless of a choice of the initial functions \( \varphi(\theta), \hat{\varphi}(\theta) \). In that case the identifiability is said to be enforced by the control input \( u(t) \).

Theorem 1 The time-delay system (1) is identifiable if and only if it is weakly controllable. Moreover, if (1) is weakly controllable then the identifiability is said to be enforced by any sufficiently nonsmooth control input \( u(t) \).

see also by the same authors: IEEE TRANSACTIONS ON AUTOMATIC CONTROL, VOL. 47, NO. 8, AUGUST 2002
On Identifiability of Linear Time-Delay Systems

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... and, more general (convolutions):

\[ P * y = Q * u, \]
\[ P * z = u, \quad y = Q * z. \]
Identification results: up to now, linear systems

- Adaptive (continuous) techniques
  - Diop, Kolmanovskii, Moraal, vanNieuwstadt – Control Eng.Pract. 9, 2001
    "Preserving stability/performance when facing an unknown time delay."
  - Orlov, Dambrine, Belkoura, Richard - IJNRC 13, 2003
    (see above)
  - Gomez, Orlov, Kolmanovskii – Automatica 43(12) 2007
    "On-line identification of SISO linear time-invariant delay systems from output measurements"

- Nonsmooth techniques (VSS)
  - Drakunov, Perruquetti, Richard, Belkoura, - Ann. Reviews in Control, 30(2) 2006
    "Delay identification in time-delay systems using variable structure observers"

- Algebraic techniques (distributions)
  - Belkoura, Richard, Fliess - Automatica 45(5) 2009
    "Parameters estimation of systems with delayed and structured entries"

The idea of algebraic estimation
(Fliess, Sira-Ramirez ESAIM COCV 2003)

Basic example, no-delay case

\[
\dot{y}(t) = ay(t) + u(t) + \gamma
\]

\[
s\ddot{y}(s) = a\ddot{y}(s) + \ddot{u}(s) + y_0 + \frac{\gamma}{s} \quad (y_0: \text{initial condition})
\]

- elimination of \(\gamma\):

\[
\frac{d}{ds}\left[s\{s\ddot{y}(s) = a\ddot{y}(s) + \ddot{u}(s) + y_0 + \frac{\gamma}{s}\}\right]
\]

\[
\Rightarrow 2s\dddot{y}(s) + s^2\dddot{y}(s) = a\left(s\dddot{y}(s) + \dddot{y}(s)\right) + s\dddot{u}(s) + \dddot{u}(s) + y_0
\]
Algebraic estim., Ctn’d

- estimation of $\alpha$: $(y_0 = 0)$

$$s^{-\nu}[2s\ddot{y}(s) + s^2\dot{y}'(s) = a(s\ddot{y}(s) + \ddot{y}(s)) + s\dot{u}(s) + \dot{u}(s)]; \quad \nu > 0$$

$$\dot{y}'(s) = \frac{dy(s)}{ds} = \mathcal{L}(-ty(t))$$

$$a = \frac{2\int_0^t d\lambda \int_0^{\lambda} y(\tau)d\tau - \int_0^t \tau y(\tau)d\tau + \int_0^t d\lambda \int_0^{\lambda} \tau u(\tau)d\tau - \int_0^t d\lambda \int_0^{\lambda} \tau u(\tau)d\tau}{\int_0^t d\lambda \int_0^{\lambda} y(\tau)d\tau - \int_0^t \tau y(\tau)d\tau}$$

$\nu = 3$

- $t$ may be very small $\rightarrow$ fast estimation
- number $\nu$ of integrations $\rightarrow$ averaging role

- parameters or states obtained via iterative integrations
  (or, more generally, low-pass filters)
- noise = fast fluctuations
Algebraic estm., Ctn’d

Same approach works for derivative estimation

\[ \dot{y}(t) + ay(t) = y(0)\delta + \gamma_0 H + bu(t - \tau) \] (basic example)

1st results: simulation

\[ y(0) = 0.3, a = 2, \tau = 0.6, \gamma_0 = 2, b = 1, u_0 = 1. \]

\[ \tau = \frac{H^2 (\dot{y}^2(t) + a\dot{y}(t))}{H^2 (\dot{y}^2(t) + a\dot{y}(t))} \quad t > \tau \]

2) real process (simple)

... and it works for delay estimation

[Belkoura, Richard, Fliess, Automatica 2009]
Algebraic estim., Ctn’d

The integrations are performed using the integration by part formula, avoiding any derivative in the algorithm.

$$\int y = -6 \int y + 6 \int (y) - 6 \int (y).$$

(6)

Remark: Filters may be used instead of integrals.

$$\tau = \frac{H^2 (\rho y^{(2)} + \alpha \rho y^{(1)})}{H^2 (\rho y^{(2)} + \alpha \rho y^{(1)})}, \quad t > \tau.$$

Partial realization scheme and simulation results.

... and may work for closed loop, variable delay estimation

$$\frac{y(s)}{e(s)} = \frac{k e^{-\tau s}}{1 + \alpha s}$$

constant parameters

slowly varying parameters
Some general references

- Niculescu (2001), Springer 
  Delay effects on stability. LNCIS Vol. 269.
- Richard (2003), Automatica (39)10 
  TDS: An overview of some recent advances and open problems"

- http://hal.inria.fr/lab/ALIEN/ 
  identification, differentiation, model-free control...

- Richard et al. (2002) Hermès 
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