Syzygies among reduction operators

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October 2, 2018

Plan

I. Motivations

- Various notions of syzygy
- Computation of syzygies

II. Reduction operators

- Linear algebra, syzygies and useless reductions
- Reduction operators and labelled reductions

III. Lattice description of syzygies

- Lattice structure of reduction operators
- Construction of a basis of syzygies
- A lattice criterion for rejecting useless reductions

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I. Motivations

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- Consider the following questions:
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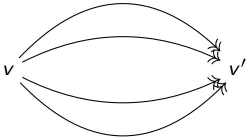

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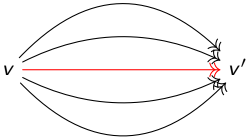

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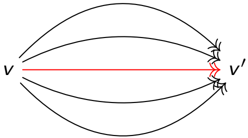

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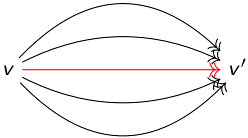

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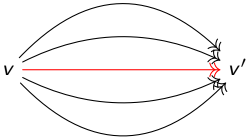

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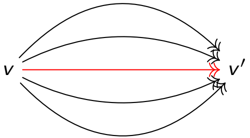

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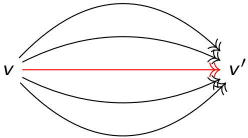

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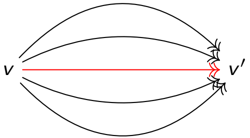

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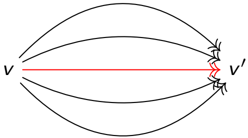

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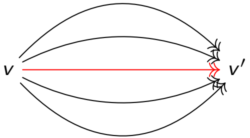

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  - Deduce a **lattice criterion** for rejecting **useless reductions** during the completion procedure.

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II. Reduction operators

Linear algebra, syzygies and useless reductions

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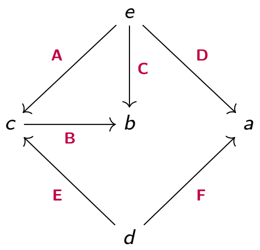

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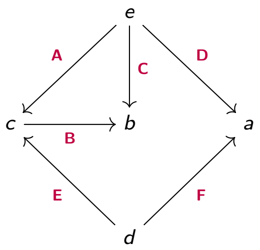

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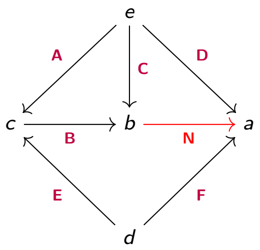

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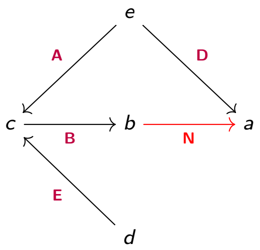

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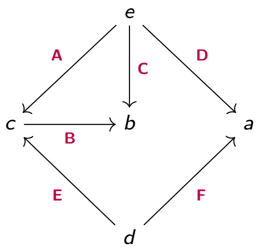

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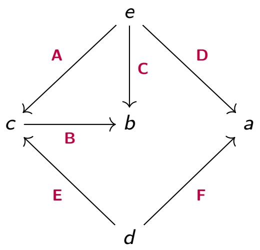

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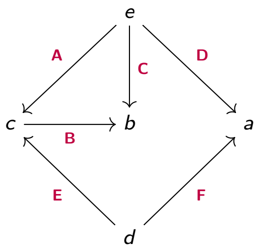

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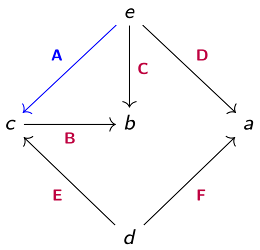

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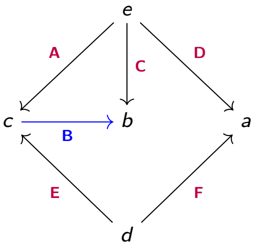

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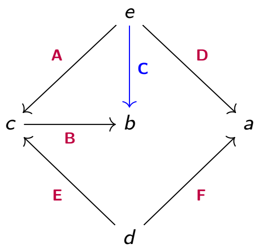

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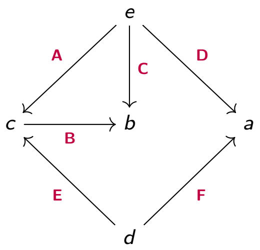

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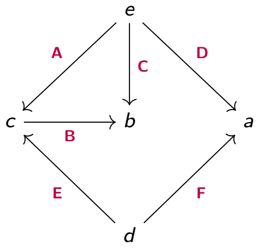

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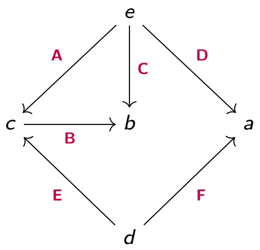

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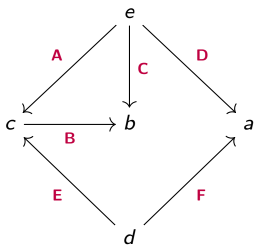

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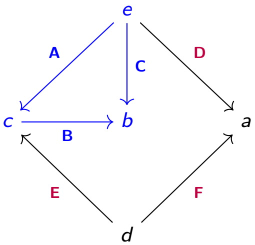

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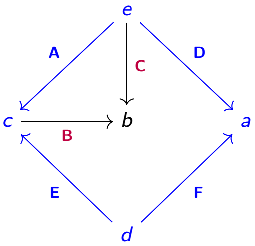

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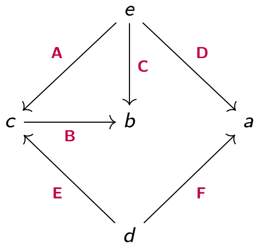

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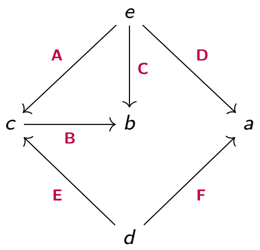

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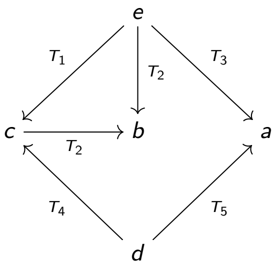

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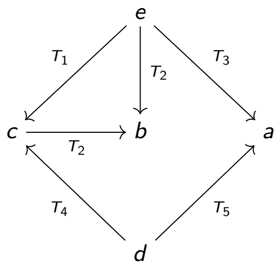

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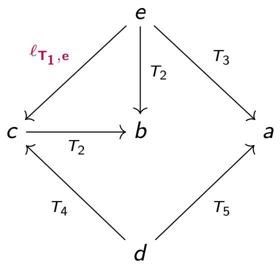

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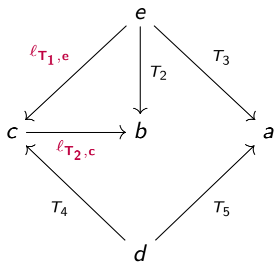

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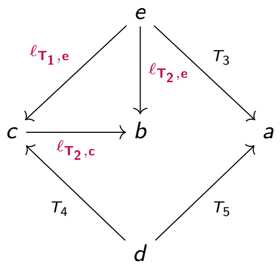

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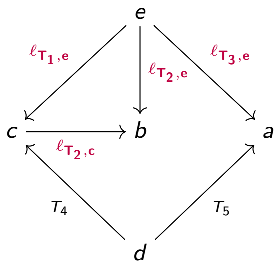

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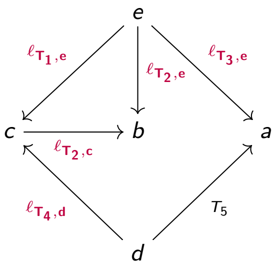

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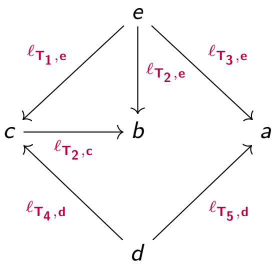

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Plan

III. Lattice description of syzygies

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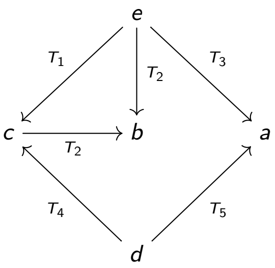

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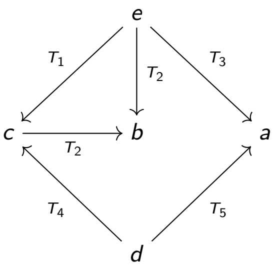

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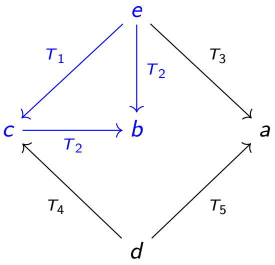

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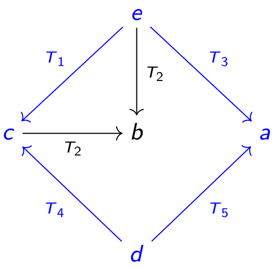

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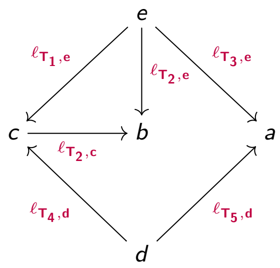

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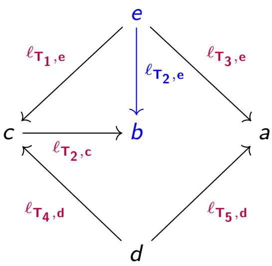

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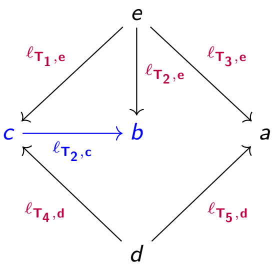

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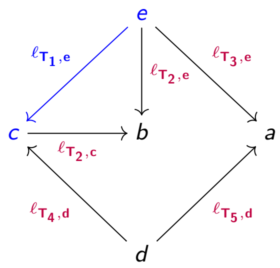

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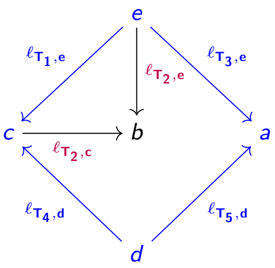

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Proposition i. *Let* $P = (T, T') \subset \mathbf{RO}(G, <)$. *We have a linear isomorphism*

$$\ker(T \vee T') \xrightarrow{\sim} \mathbf{syz}(P).$$

Proposition ii. *Let* $F = \{T_1, \cdots, T_n\} \subset \mathbf{RO}(G, <)$. *For every integer* $2 \leq i \leq n$, *we have a short exact sequence*

$$0 \longrightarrow \mathbf{syz}(T_1, \cdots, T_{i-1}) \xrightarrow{\iota_i} \mathbf{syz}(T_1, \cdots, T_i) \xrightarrow{\pi_i} \mathbf{syz}(T_1 \wedge \cdots \wedge T_{i-1}, T_i) \longrightarrow 0.$$

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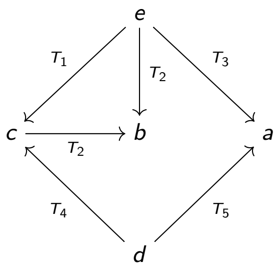

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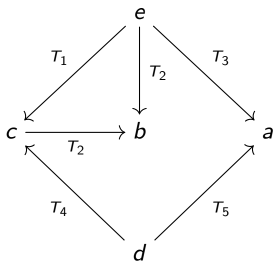

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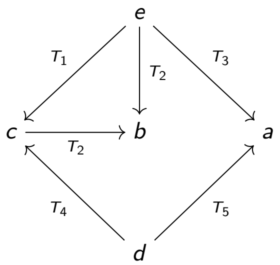

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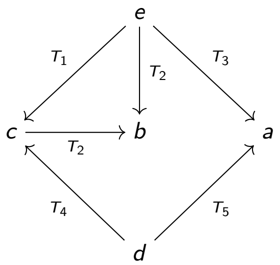

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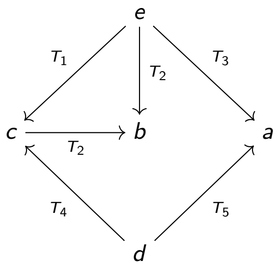

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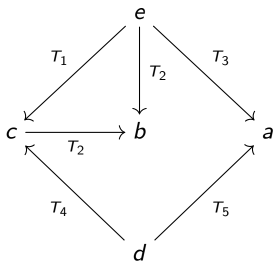

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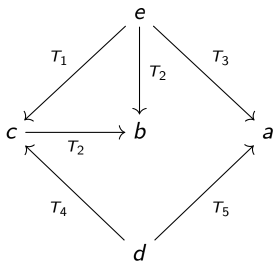

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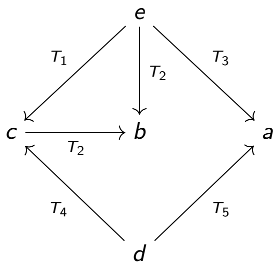

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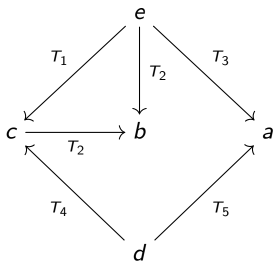

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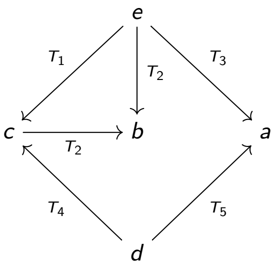

Step 3. $\ker((T_1 \wedge T_2 \wedge T_3) \vee T_4) = \{0\}$.

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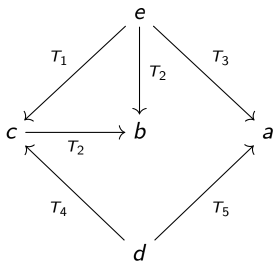

Step 4. We have $\ker((T_1 \wedge T_2 \wedge T_3 \wedge T_4) \vee T_5) = \mathbb{K}\{d - a\}$.

- $d - a = (d - T_4(d)) + (e - T_3(e)) - (e - T_1(e))$.
- $d - a = d - T_5(d)$.
- We get the second basis element:

$$s_2 = u_{5,d} - u_{4,d} - u_{3,e} + u_{1,e}.$$

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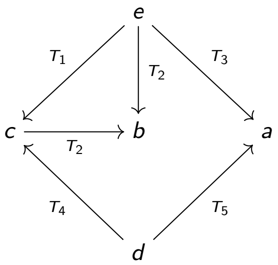

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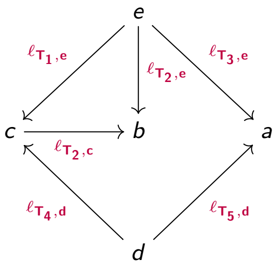

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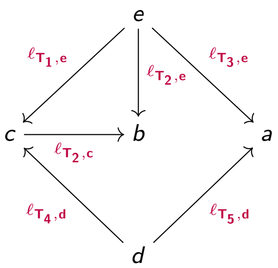

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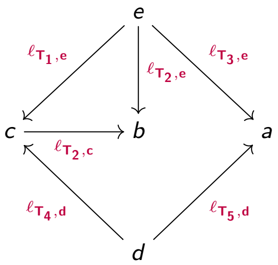

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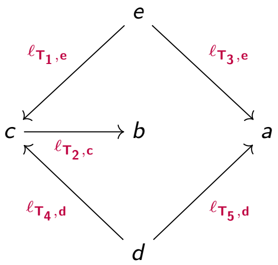

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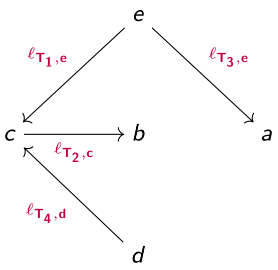

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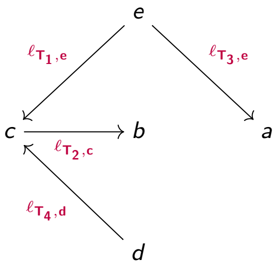

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THANK YOU FOR YOUR ATTENTION!