Reduction operators: completion, syzygies and Koszul duality

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February 19, 2019

Plan

I. Motivations

- ▷ Computational problems and rewriting theory
- ▷ Termination, confluence and Gröbner bases

II. Reduction operators

- ▷ Reduction operators and linear rewriting systems
- ▷ Lattice structure of reduction operators
- ▷ Lattice descriptions of confluence and completion

III. Applications

- ▷ Lattice structure and linear basis of syzygies
- ▷ Construction of a contracting homotopy for the Koszul complex

IV. Conclusion and current works

I. MOTIVATIONS

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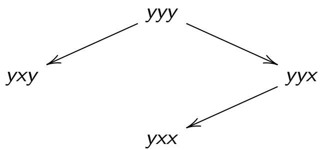

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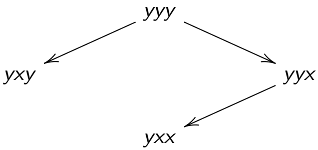

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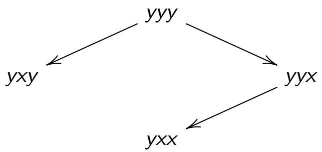

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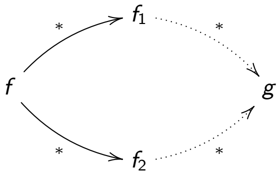

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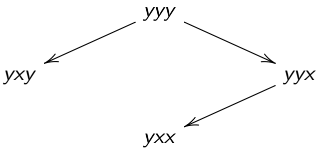

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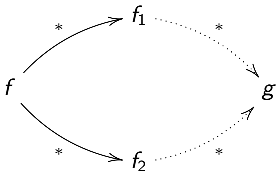

- If the orientation is terminating and confluent, normal monomials form a linear basis

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  - Formalisation of noncommutative GB [Gergman 1978]
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- **Purpose of the talk:** extend the functional approach
  - Lattice characterisation of the confluence property (for abstract linear rewriting systems)
  - Lattice interpretations of the completion procedure and syzygies
  - Application to (nonquadratic) Koszul duality

II. REDUCTION OPERATORS

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so that

$$v \underset{T_1}{\longrightarrow} (v_1 + v_2,\ 0,\ v_3 + v_4,\ 0) \quad \text{and} \quad v \underset{T_2}{\longrightarrow} (v_1,\ v_2 + v_4,\ v_3,\ 0)$$

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$$\begin{aligned} \ker(T_1 \wedge T_2) &= \mathbb{K}\langle g_2 - g_1,\ g_4 - g_3,\ g_4 - g_2 \rangle \\ &= \mathbb{K}\langle \mathbf{g}_2 - \mathbf{g}_1,\ \mathbf{g}_3 - \mathbf{g}_1,\ \mathbf{g}_4 - \mathbf{g}_1 \rangle \end{aligned}$$

  - ker being a bijection:

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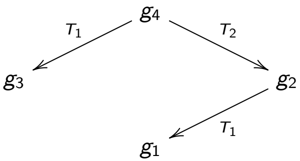

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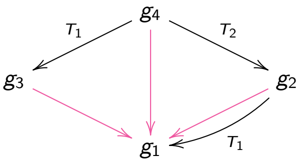

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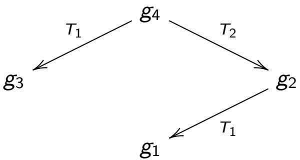

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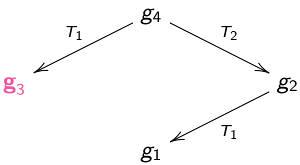

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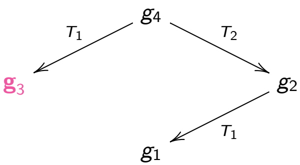

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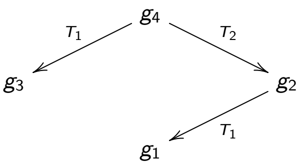

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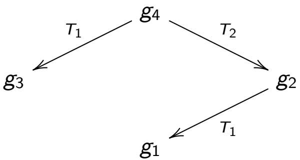

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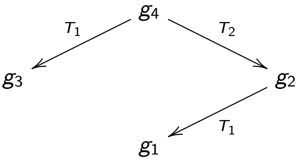

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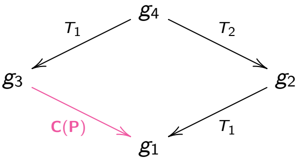

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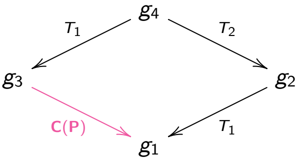

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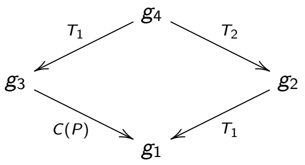

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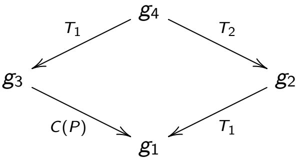

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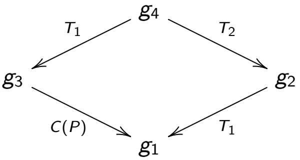

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- **Theorem.** *We have*

$$C(F) = (\wedge F) \vee \left(\vee \overline{F}\right)$$

III. APPLICATIONS

- Computation of syzygies appear in
  - **Completion procedures:** remove useless reductions/critical pairs
  - **Higher-dimensional algebra:** compute homological/homotopical invariants
  - **Standardisation problems:** choose a standard rewriting path (e.g. Janet bases)

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  - *There is an isomorphism:* $\ker(T \vee T') \xrightarrow{\sim} \mathbf{syz}(T, T'), \ v \mapsto (-v, v)$

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- A linear basis of $\mathbf{syz}(F)$ is obained as follows
  - By induction using $\mathbf{syz}(F_{i-1}) \subseteq \mathbf{syz}(F_i)$
  - A supplement is constructed using $\mathbf{syz}(F_i)/\mathbf{syz}(F_{i-1}) \simeq \ker((\wedge F_{i-1}) \vee T_i)$

- **Notation.** $\ell_{i,g} \ := \ (0,\cdots, \ 0, \ g - T_i(g), \ 0,\cdots, \ 0) \ \in \ \ker(T_1) \times \cdots \times \ker(T_n)$

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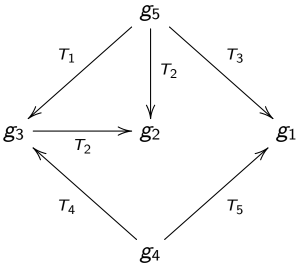

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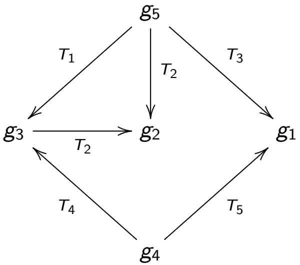

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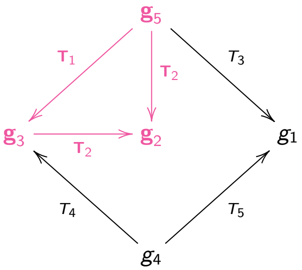

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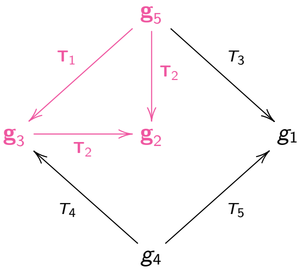

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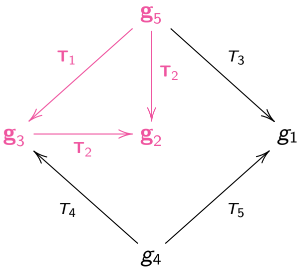

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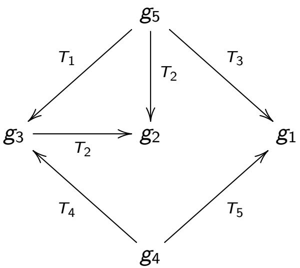

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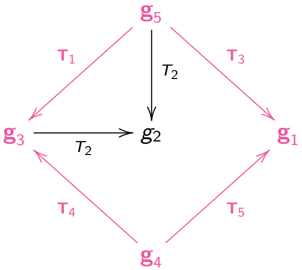

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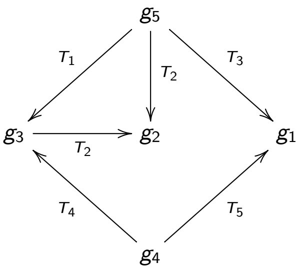

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  - **Output.** $\mathbf{syz}(F) = \mathbb{K}\langle s_1, s_2 \rangle$

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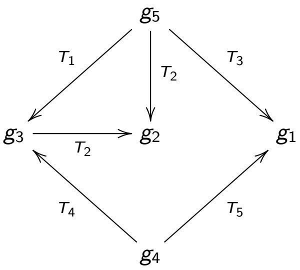

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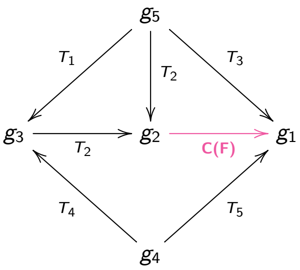

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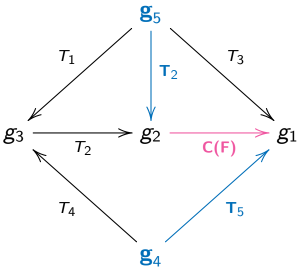

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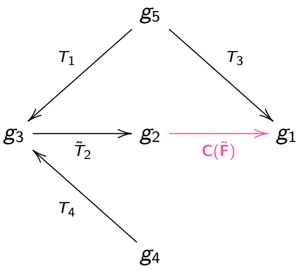

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IV. CONCLUSION AND CURRENT WORKS

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THANK YOU FOR YOUR ATTENTION!