Quotients of the magmatic operad: lattice structures and convergent rewrite systems

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Plan

I. Motivations

- ▷ Motivating example: an oscillating Hilbert series
- ▷ Nonsymmetric operads
- ▷ Presentations and Gröbner bases for operads

II. Magmatic quotients

- ▷ The category of magmatic quotients
- ▷ The lattice of magmatic quotients
- ▷ A Grassmann formula analog

III. Comb associative operads

- ▷ Definition of CAs operads
- ▷ The lattice of CAs operads
- ▷ Completion of CAs operads

IV. Conclusion and perspectives

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- Hilbert series may be computed using Gröbner bases, that are terminating and confluent rewrite systems;
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- Is the operad $\mathbf{CAs}^{(3)}$ presented by a finite Gröbner basis?
  - Yes: using the Buchberger/Knuth-Bendix's completion procedure.

- A **nonsymmetric linear operad** is a positively graded ($\mathbb{K}$-)vector space

$$\mathscr{O} = \bigoplus_{n \in \mathbb{N}} \mathscr{O}(n),$$

together with

- a distinguished element $\mathbf{1} \in \mathscr{O}(1)$;
- partial compositions $\circ_i : \mathscr{O}(n) \otimes \mathscr{O}(m) \to \mathscr{O}(n+m-1)$, $\forall 1 \leq i \leq n$;

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- How to construct operads?
  - Using **presentations** by **generators and relations** $\langle \mathscr{X} \mid \mathscr{R} \rangle$.

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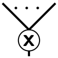

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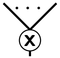
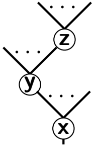
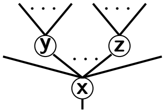


  - $\mathscr{F}(\mathscr{X}) := \{$linear combinations of syntactic trees$\}$





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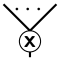

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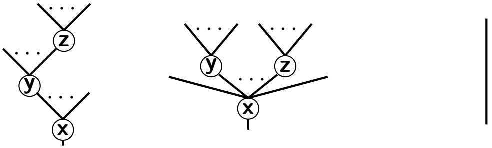

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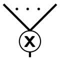

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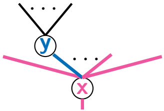

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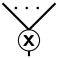

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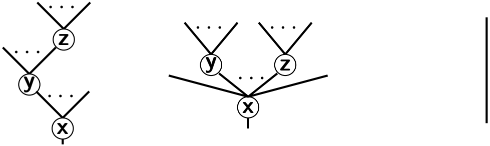

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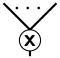

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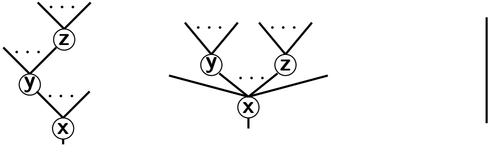

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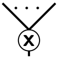

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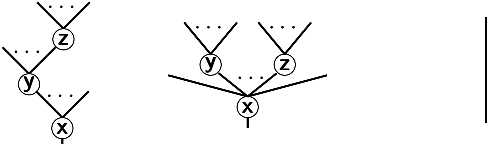

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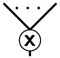

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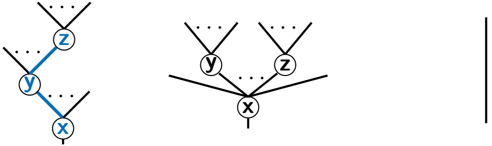

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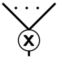

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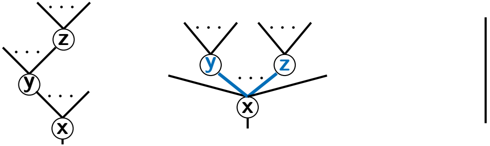

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  - commutativity of parallel compositions: $(\mathbf{x} \circ_i \mathbf{y}) \circ_{j+m-1} \mathbf{z} = (\mathbf{x} \circ_j \mathbf{z}) \circ_i \mathbf{y}$, where $i < j$ and m is the arity of $\mathbf{y}$.

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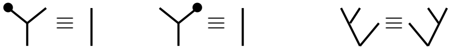

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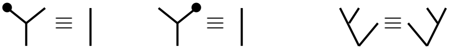

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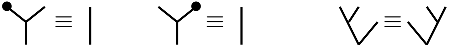

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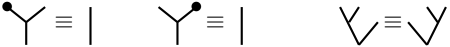

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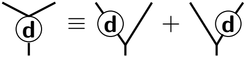

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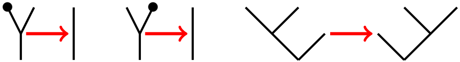

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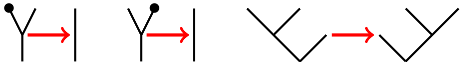

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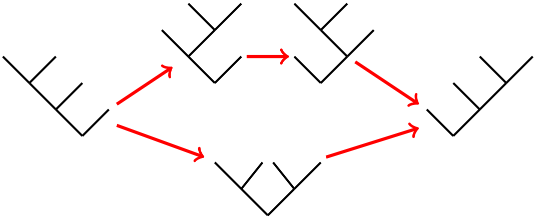

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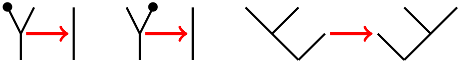

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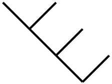

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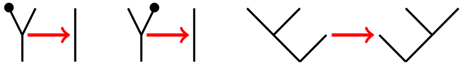

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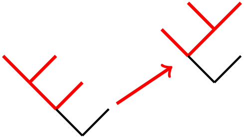

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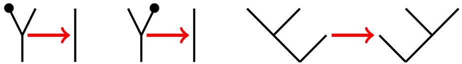

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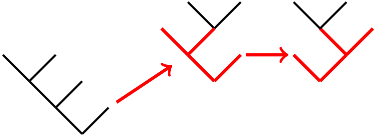

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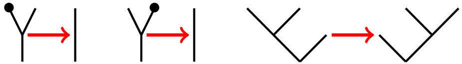

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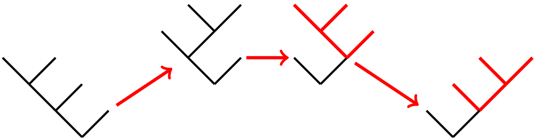

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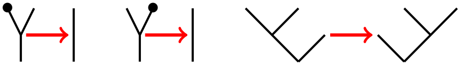

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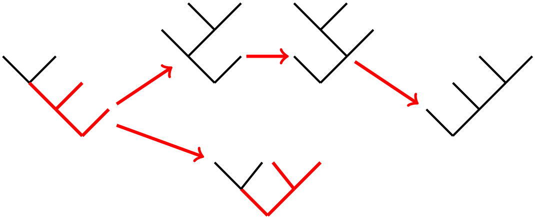

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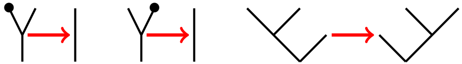

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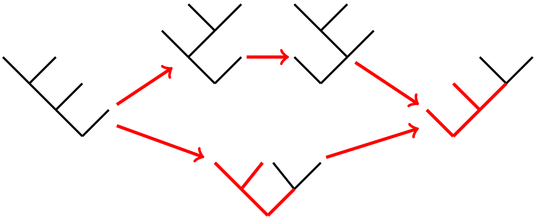

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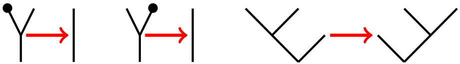

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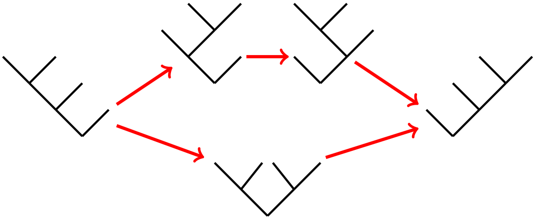

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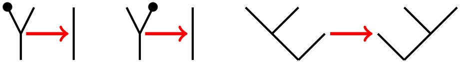

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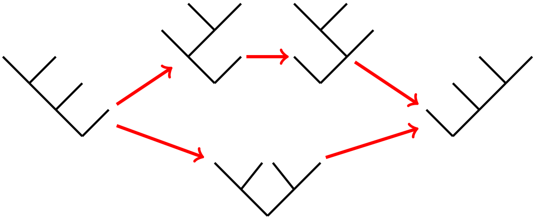

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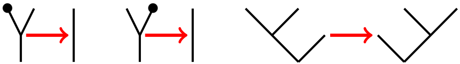

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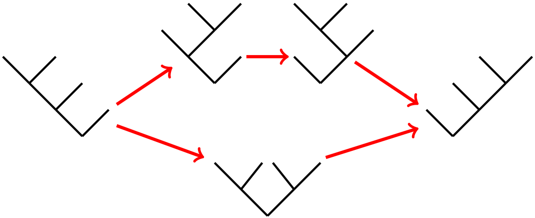

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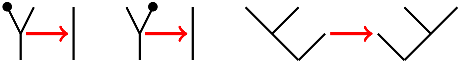

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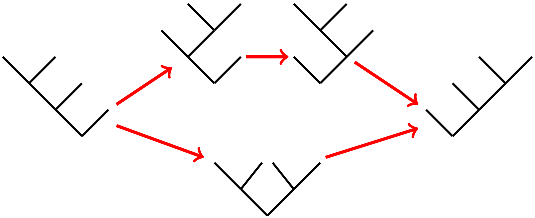

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- Gröbner bases are computed by the Buchberger/Knuth-Bendix's completion procedure.

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- We study the induced poset on $\mathbf{CAs}$:
  - we present **new** lattice operations on this poset;
  - we study the existence of finite Gröbner bases for $\mathbf{CAs}^{(\gamma)}$ operads;
  - we deduce the complete expression of the Hilbert series of $\mathbf{CAs}^{(3)}$.

Plan

II. Lattice of magmatic quotients

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- A magmatic quotient is a quotient operad $\mathscr{O} = \mathbb{K}\mathbf{Mag}/_I$.
  - ▷ Alternatively: it is an operad over one binary generator $[\star]_I$.
  - ▷ $\mathcal{Q}(\mathbb{K}\mathbf{Mag}) := \{\text{magmatic quotients}\}$.

Lemma. *Given* $\mathscr{O}_1 = \mathbb{K}\mathbf{Mag}/_{I_1}$ *and* $\mathscr{O}_2 = \mathbb{K}\mathbf{Mag}_{I_2}$, *we have* $\dim(\text{Hom}(\mathscr{O}_1, \mathscr{O}_2)) \leq 1$.

Sketch of proof. Let $\varphi \in \text{Hom}(\mathscr{O}_1, \mathscr{O}_2)$,

- ▷ taking arities into account: $\exists \lambda \in \mathbb{K}$, s.t. $\varphi([\star]_{I_1}) = \lambda[\star]_{I_2}$;
- ▷ by the universal property of the quotient: if $\lambda \neq 0$, then $I_1 \subseteq I_2$;
- ▷ if $I_1 \subseteq I_2$, then $\forall \mu \in \mathbb{K} \setminus \{0\}$, $\exists \psi \in \text{Hom}(\mathscr{O}_1, \mathscr{O}_2)$ s.t. $\psi([\star]_{I_1}) = \mu[\star]_{I_2}$.

Remark. A nonzero operad morphism between magmatic quotients is surjective.

- Let $\mathscr{O}_1 = \mathbb{K}\mathbf{Mag}/_{I_1}$ and $\mathscr{O}_2 = \mathbb{K}\mathbf{Mag}/_{I_2}$;
  - we have $\dim(\mathrm{Hom}(\mathscr{O}_1, \mathscr{O}_2)) \leq 1$;
  - $\dim(\mathrm{Hom}(\mathscr{O}_1, \mathscr{O}_2)) = 1$ iff $I_1 \subseteq I_2$;
  - $\dim(\mathrm{Hom}(\mathscr{O}_1, \mathscr{O}_2)) = 1$ iff $\exists \varphi : \mathscr{O}_1 \to \mathscr{O}_2$ surjective.

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- Let $\preceq_{\mathrm{i}} \subsetneq \mathcal{Q}(\mathbb{K}\mathbf{Mag}) \times \mathcal{Q}(\mathbb{K}\mathbf{Mag})$ defined by
  - $\mathscr{O}_2 \preceq_{\mathrm{i}} \mathscr{O}_1$ iff $\dim(\mathrm{Hom}(\mathscr{O}_1, \mathscr{O}_2)) = 1$;

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  - $\mathcal{O}_1 \wedge_{\mathrm{i}} \mathcal{O}_2 = \mathbb{K}\mathbf{Mag}/_{I_1 + I_2}$;
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Theorem [C.-Cordero-Giraudo, 2018]. *Consider the notations introduced above.*

i. *The tuple* $(\mathcal{Q}(\mathbb{K}\mathbf{Mag}), \preceq_{\mathrm{i}}, \wedge_{\mathrm{i}}, \vee_{\mathrm{i}})$ *is a lattice.*

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Theorem [C.-Cordero-Giraudo, 2018]. *Consider the notations introduced above.*

i. *The tuple* $(\mathcal{Q}(\mathbb{K}\mathbf{Mag}), \preceq_{\mathrm{i}}, \wedge_{\mathrm{i}}, \vee_{\mathrm{i}})$ *is a lattice.*

ii. *We have the following Grassmann formula analog:*

$$\mathcal{H}_{\mathcal{O}_1 \vee_{\mathrm{i}} \mathcal{O}_2}(t) + \mathcal{H}_{\mathcal{O}_1 \wedge_{\mathrm{i}} \mathcal{O}_2}(t) = \mathcal{H}_{\mathcal{O}_1}(t) + \mathcal{H}_{\mathcal{O}_2}(t).$$

- Let **As** $:= \mathbb{K}\mathbf{Mag}/_{I_{\mathbf{As}}}$ and **AAs** $:= \mathbb{K}\mathbf{Mag}/_{I_{\mathbf{AAs}}}$, where $I_{\mathbf{As}}$ and $I_{\mathbf{AAs}}$ are generated by

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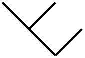
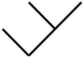
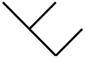
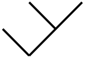
 $-$  and  $+$ 

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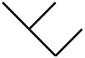
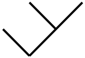
 $-$  and  $+$ 

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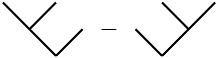

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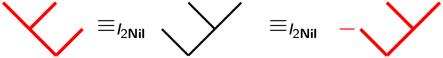

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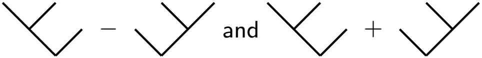

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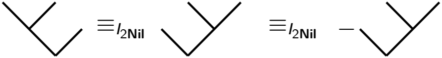

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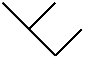
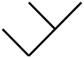
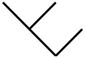
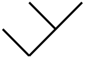
 $-$  and  $+$ 

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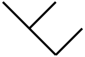 $\equiv_{I_{2\mathbf{Nil}}}$  $\equiv_{I_{2\mathbf{Nil}}}$ $-$ 

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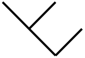 and 

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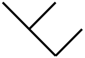
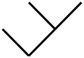
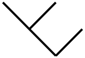
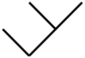
 $-$  and  $+$ 

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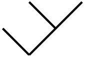
 $\equiv_{I_{2\mathbf{Nil}}}$  $\equiv_{I_{2\mathbf{Nil}}}$ $-$ 

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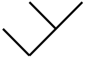
 and 

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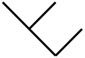
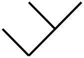
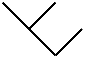
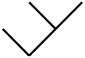
 $-$  and  $+$ 

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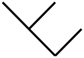
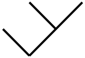
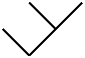
 $\equiv_{I_{2\mathbf{Nil}}}$  $\equiv_{I_{2\mathbf{Nil}}}$ $-$ 

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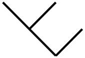
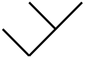
 and 

- Letting $I_{\mathbb{K}\mathbf{RC}^{(3)}} := \{\mathbf{x} - \mathbf{y} \mid \mathbf{x}$ and $\mathbf{y}$ are trees of arity 4$\}$, we have $\mathbb{K}\mathbf{RC}^{(3)} = \mathbf{As} \vee_{\mathrm{i}} \mathbf{AAs}$;

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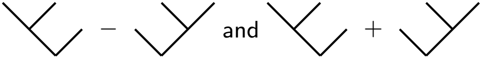

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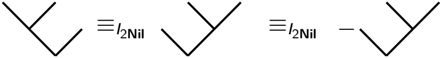

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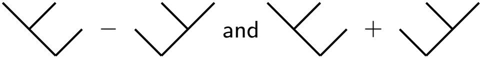

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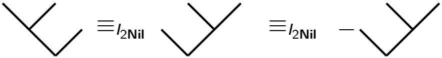

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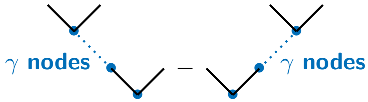
  - using the Grassmann formula, one shows that π is an isomorphism.

Plan

III. Comb associative operads

- $\gamma \geq 1$: a positive integer;
  - $I_{\mathbf{CAs}(\gamma)}$: the ideal generated by



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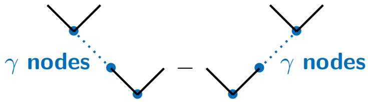

  - $\mathbf{CAs}^{(\gamma)} := \mathbf{Mag}/_{I_{\mathbf{CAs}^{(\gamma)}}}$ is called the **γ-comb associative operad**.

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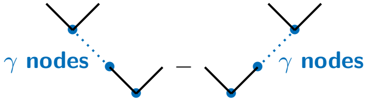

  - $\mathbf{CAs}^{(\gamma)} := \mathbf{Mag}/_{I_{\mathbf{CAs}(\gamma)}}$ is called the γ-comb associative operad.

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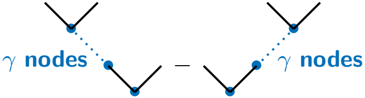

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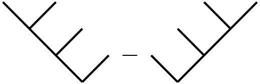

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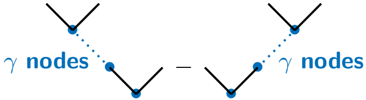

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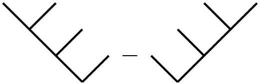

- Objective of the section: show that

$$\mathbf{CAs} := \left\{ \mathbf{CAs}^{(\gamma)} \mid \gamma \geq 1 \right\}$$

admits a lattice structure.

- $\preceq_{\mathrm{d}}$: the restriction of $\preceq_{\mathrm{i}}$ to **CAs**;
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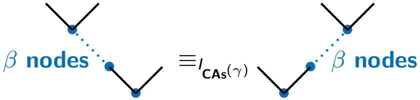

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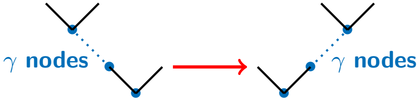

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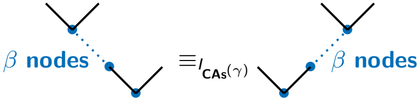

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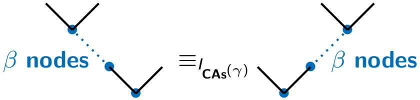

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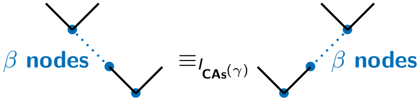

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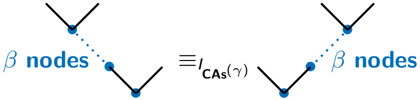

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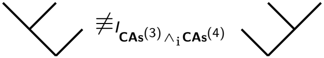

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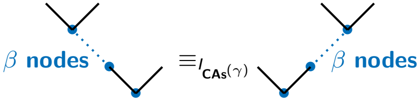

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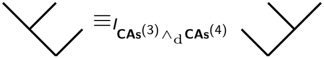

since $\mathbf{CAs}^{(3)} \wedge_{\mathrm{d}} \mathbf{CAs}^{(4)} = \mathbf{CAs}^{(\mathrm{gcd}(2,3)+1)} = \mathbf{CAs}^{(2)} = \mathbf{As}$.

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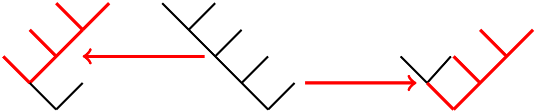

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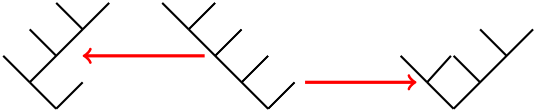

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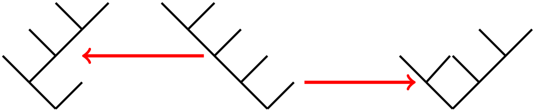

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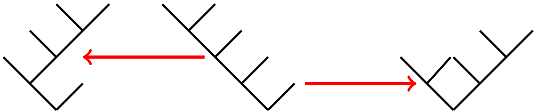

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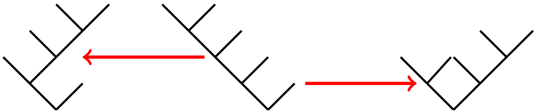

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value of n	2	3	4	5	6	7	8	9	10
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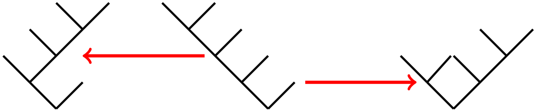

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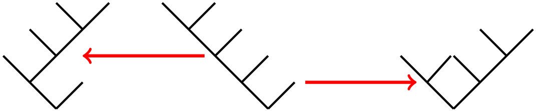

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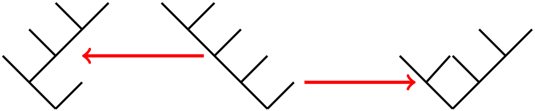

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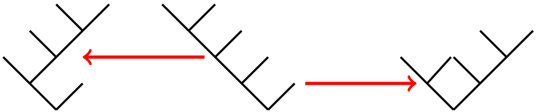

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Plan

IV. Conclusion and perspectives

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  - we equipped $\mathcal{Q}(\mathbb{K}\textbf{Mag})$ with a lattice structure and provide a Grassmann formula analog;
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THANK YOU FOR LISTENING!