Confluence algebras and acyclicity of the Koszul complex

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1 Introduction

We are concerned with computing resolutions of associative algebras over a field $\mathbb{K}$ using techniques from rewriting theory. Several resolutions of algebras are based on convergent presentations (see [Ani86] or [Kob90]). Here, we study *N-homogeneous algebras*, that is algebras admitting an N-homogeneous presentation $\langle X \mid R\rangle$. We denote by $\mathbb{K}X$ and $\overline{R}$ the vector space spanned by X and the sub vector space of $\mathbb{K}X^{\otimes N}$ spanned by R, respectively. We consider the function $l_N : \mathbb{N} \longrightarrow \mathbb{N}$ defined by:

$$l_N(n) = \begin{cases} kN, & \text{if } n = 2k, \\ kN+1, & \text{if } n = 2k+1. \end{cases}$$

Given an N-homogeneous algebra A, we try to compute a resolution of $\mathbb{K}$ in the following way:

$$\cdots \xrightarrow{\partial_{n+1}} A \otimes J_n^N \xrightarrow{\partial_n} A \otimes J_{n-1}^N \longrightarrow \cdots \xrightarrow{\partial_4} A \otimes J_3^N \xrightarrow{\partial_3} A \otimes \overline{R} \xrightarrow{\partial_2} A \otimes \mathbb{K}X \xrightarrow{\partial_1} A \xrightarrow{\varepsilon} \mathbb{K} \longrightarrow 0, \quad (1)$$

where, for every integer n such that $n \geq 3$, we have:

$$J_n^N = \bigcap_{i=0}^{l_N(n)-N} \mathbb{K}X^{\otimes i} \otimes \overline{R} \otimes \mathbb{K}X^{\otimes l_N(n)-N-i}.$$

In general, the complex (1), called the *Koszul complex of* A, is not acyclic. The algebras for which this complex is cyclic are *N-Koszul algebras.*

The N-Koszul algebras have been defined by Berger in [Ber01]. This notion generalises the one of *Koszul algebra* defined by Priddy (see [Pri70]). The N-Koszul algebras have been studied through computational approaches based a monomial order, that is, a well founded total order on the set of monomials. In [Ani86], Anick used Gröbner basis to construct a free resolution of $\mathbb{K}$. This resolution enables us to conclude that an algebra which admits a quadratic Gröbner basis is Koszul. In [Ber01], Berger studied N-homogeneous algebras with a *side-confluent presentation* [1]. More precisely, we can associate with any N-homogeneous presentation $\langle X \mid R\rangle$ of A such that X is finite, an unique linear projector S of $\mathbb{K}X^{\otimes N}$ (the formal definition of this operator will be given in Section 2). This projector maps any element of $\mathbb{K}X^{\otimes N}$ to a better one with respect to the monomial order. The presentation

[1] This notion corresponds to the one of *X-confluent algebra* in [Ber01]. However, we prefer to use our terminology because the property of confluence depends on the presentation.

$\langle X \mid R\rangle$ is said to be side-confluent if for every integer m such that $N+1 \leq m \leq 2N-1$, there exists an integer k which satisfies:

$$\langle S \otimes \mathrm{Id}_{\mathbb{K}X^{\otimes m-N}}, \mathrm{Id}_{\mathbb{K}X^{\otimes m-N}} \otimes S\rangle^k = \langle \mathrm{Id}_{\mathbb{K}X^{\otimes m-N}} \otimes S, S \otimes \mathrm{Id}_{\mathbb{K}X^{\otimes m-N}}\rangle^k,$$

where $\langle t, s\rangle$ denotes the product $\cdots sts$ with k factors. The algebra presented by:

$$\left\langle s_1, s_2 \mid \langle s_1, s_2\rangle^k = \langle s_2, s_1\rangle^k,\ s_i^2 = s_i,\ i = 1, 2\right\rangle,$$

is naturally associated with a side-confluent presentation. This algebra is the *confluence algebra of degree k*. In [Ber98, Section 5], Berger used specific representations of these algebras to construct a contracting homotopy for the Koszul complex of a quadratic algebra which admits a side-confluent presentation. This construction enables us to conclude that a quadratic algebra admitting a side-confluent presentation is Koszul. However, when N is greater than 2, an N-homogeneous algebra admitting a side-confluent presentation is not necessarily N-Koszul. Indeed, such an algebra is N-Koszul if and only if the *extra-condition* holds (see [Ber01, Proposition 3.4]). The extra-condition is stated as follows:

$$(ec): \quad \left(\mathbb{K}X^{\otimes m} \otimes \overline{R}\right) \cap \left(\overline{R} \otimes \mathbb{K}X^{\otimes m}\right) \subset \mathbb{K}X^{\otimes m-1} \otimes \overline{R} \otimes \mathbb{K}X, \text{ for every } 2 \leq m \leq N-1.$$

We remark, that when N is equal to 2, the extra-condition is an empty condition.

We deduce of the works from [Ber01] that the Koszul complex of an algebra A admitting a side-confluent presentation satisfying the extra-condition is acyclic. However, there does not exist an explicit contracting homotopy for the Koszul complex of A. In [Che16] we construct such a contracting homotopy. Our construction uses representations of confluence algebras. However, we think that we can obtain the same construction using techniques from higher-dimensional rewriting theory. More precisely, given a convergent presentation of an N-homogeneous algebra A satisfying the extra-condition, we could construct a polygraphic resolution of A such that the set of n-cells of this polygraph is a basis of the vector space J_n^N.

2 Reduction operators and confluence algebras

Our construction uses *reduction operators* and *confluence algebras*. This section consists in an overview on these notions. We fix a totally ordered finite set Y.

2.1. Reduction operators. A linear projector T of $\mathbb{K}Y$ is called a *reduction operator relatively to* Y if for every $y \in Y$, we have either $T(y) = y$ or $T(y) < y$. We denote by $\mathrm{Red}\,(Y)$ the set of reduction operators relatively to Y.

2.2. Lattice structure. The set $\mathrm{Red}\,(Y)$ admits a lattice structure. To define it, recall from [Ber98, Theorem 2.3] that the map

$$\begin{aligned} \theta_Y \colon \mathrm{Red}\,(Y) &\longrightarrow \mathscr{L}(\mathbb{K}Y), \\ T &\longmapsto \ker(T) \end{aligned}$$

is a bijection. The order over $\mathrm{Red}\,(Y)$ is defined by:

$$T_1 \preceq T_2, \text{ if } \ker(T_2) \subset \ker(T_1).$$

The lower bound $T_1 \wedge T_2$ and the upper bound $T_1 \vee T_2$ of two elements T_1 and T_2 of $\mathrm{Red}\,(Y)$ are defined in the following way:

$$\begin{cases} T_1 \wedge T_2 = \theta_Y^{-1}\left(\ker(T_1) + \ker(T_2)\right), \\ T_1 \vee T_2 = \theta_Y^{-1}\left(\ker(T_1) \cap \ker(T_2)\right). \end{cases}$$

2.3. Confluent pairs of reduction operators. A pair $P = (T_1, T_2)$ of reduction operators relatively to Y is said to be *confluent* if there exists an integer k such that:

$$\langle T_1, T_2\rangle^k = \langle T_2, T_1\rangle^k .$$

2.4. Confluence algebras. Let k be an integer. The *confluence algebra of degree* k is the algebra presented by

$$\left\langle s_1, s_2 \mid \ s_i^2 = s_i,\ \langle s_1, s_2\rangle^k = \langle s_2, s_1\rangle^k ,\ i = 1,2 \right\rangle .$$

2.5. Reduction operators and N-homogeneous presentations. We end this section giving the link between reduction operators and N-presentations. Let A be an N-homogeneous algebra and let $\langle X \mid R\rangle$ be an N-homogeneous presentation of A such that X is finite. In particular, the set $X^{(N)}$ of words of length N is finite. We let:

$$S = \theta^{-1}_{X^{(N)}}\left(\overline{R}\right).$$

The presentation $\langle X \mid R\rangle$ is said to be *side-confluent* if for every integer m such that $N+1 \leq m \leq 2N-1$, the pair $(S \otimes \mathrm{Id}_{\mathbb{K}X^{\otimes m-N}}, \mathrm{Id}_{\mathbb{K}X^{\otimes m-N}} \otimes S)$ is confluent.

2.6. Extra-confluent presentations. Let A be an N-homogeneous algebra. A side-presentation $\langle X \mid R\rangle$ such that X is finite and the extra-condition holds is said to be *extra-confluent*.

3 The contracting homotopy

We present the different steps of our construction. We fix an N-homogeneous algebra together with an N-homogeneous presentation $\langle X \mid R\rangle$ of A. We assume that X is finite and totally ordered. In particular, for everey integer m, the set $X^{(m)}$ is finite and totally ordered. We write $V = \mathbb{K}X$ and for every integer m we identity $V^{\otimes m}$ to $\mathbb{K}X^{(m)}$.

3.1. Reduction pairs associated with a presentation. For every integers n and m such that $m \geq l_N(n)$, we consider the following reduction operators:

$$F_1^{n,m} = \theta^{-1}_{X^{(m)}}\left(I(R)_{m-l_N(n)} \otimes V^{\otimes l_N(n)}\right),$$

$$F_2^{n,m} = \begin{cases} \mathrm{Id}_{V^{\otimes m}}, \text{ if } m < l_N(n+1), \\ \theta^{-1}_{X^{(m)}}\left(V^{\otimes m-l_N(n+1)} \otimes J_{n+1}\right), \text{ otherwise.} \end{cases}$$

The pair $(F_1^{n,m}, F_2^{n,m})$ is denoted by $P_{n,m}$ and is called the *reduction pair of bi-degree* (n,m) associated with $\langle X \mid R\rangle$. We point the fact that the finiteness condition over X is necessary to define these pairs. The lattice structure on the set of reduction operators enables us to prove:

3.2. Theorem. *Let A be an N-homogeneous algebra admitting a side-confluent presentation $\langle X \mid R\rangle$, where X is a finite set. The reduction pairs associated with $\langle X \mid R\rangle$ are confluent.*

3.3. The left bound. The reduction pairs associated with a side-confluent presentation $\langle X \mid R\rangle$ enable us to define a family of representations of confluence algebras in the following way:

$$\begin{aligned} \varphi^{P_{n,m}} \colon \left\langle s_1, s_2 \mid \langle s_1, s_2\rangle^{k_{n,m}} = \langle s_2, s_1\rangle^{k_{n,m}} ,\ s_i^2 = s_i,\ i = 1,2 \right\rangle &\longrightarrow \mathrm{End}\left(V^{\otimes m}\right), \\ s_i &\longmapsto F_i^{n,m} \end{aligned}$$

where the integer $k_{n,m}$ satisfies:

$$\langle F_1^{n,m}, F_2^{n,m} \rangle^{k_{n,m}} = \langle F_2^{n,m}, F_1^{n,m} \rangle^{k_{n,m}}.$$

For every integers n and m we consider the following specific element of $\mathscr{A}_{k_{n,m}}$:

$$\gamma_1 = (1 - s_2)\left(s_1 + s_1 s_2 s_1 + \cdots + \langle s_2, s_1 \rangle^{2i+1}\right),$$

where the integer i depends on $k_{n,m}$. The elements $\varphi^{P_{n,m}}(\gamma_1)$ are used to construct a family of $\mathbb{K}$-linear maps

$$\begin{aligned} &h_0 : A \longrightarrow A \otimes V, \\ &h_1 : A \otimes V \longrightarrow A \otimes \overline{R}, \\ &h_2 : A \otimes \overline{R} \longrightarrow A \otimes J_3^N, \\ &h_n : A \otimes J_n^N \longrightarrow A \otimes J_{n+1}^N, \text{ for } n \geq 3. \end{aligned}$$

We give more details about the constructions of these maps. The confluence property implies that every element $f \in \mathbb{K}\langle X \rangle$ admits a unique normal form for $\langle X \mid R \rangle$, denoted by $\widehat{f}$. Then, we have a $\mathbb{K}$-linear isomorphism:

$$\begin{aligned} \phi\colon A &\longrightarrow N, \\ \overline{f} &\longmapsto \widehat{f} \end{aligned}$$

where N is the vector space of normal forms for $\langle X \mid R \rangle$. It turns out that for every integer n, the image of the $\mathbb{K}$-linear map $h'_n : \bigoplus_{m \geq l_N(n)} V^{\otimes m} \longrightarrow T(V)$ defined by

$$h'_{n|V^{\otimes m}} = \varphi^{P_{n,m}}(\gamma_1),$$

is included in $N \otimes J_{n+1}$. Then, for every integer n, the map $h_n : A \otimes J_n \longrightarrow A \otimes J_{n+1}$ is defined by:

$$h_n = \phi_{n+1}^{-1} \circ h'_n \circ \phi_n : A \otimes J_n \longrightarrow A \otimes J_{n+1},$$

where ϕ_n is equal to $\phi \otimes \mathrm{Id}_{V^{\otimes l_N(n)}}$. The family (h_n) is called, the *left bound of* $\langle X \mid R \rangle$.

3.4. Reduction relations. In general, the the left bound of a side-confluent presentation is not a contracting homotopy for the Koszul complex of A. In fact, this property is true if and only if for every integers n and m such that $m \geq l_N(n)$, we have:

$$(r_{n,m}) \quad F_1^{n,m} \wedge F_2^{n,m}{}_{|K_n^{(m)}} = F_1^{n-1,m} \vee F_2^{n-1,m}{}_{|K_n^{(m)}},$$

where $K_n^{(m)}$ is the vector space $\left(N \cap V^{\otimes m - l_N(n)}\right) \otimes J_n$. When the relation $(r_{n,m})$ hold, we say that $\langle X \mid R \rangle$ satisfy the *reduction relations.* Then, we have the following:

3.5. Proposition. *Let A be an N-homogeneous algebra. Assume that A admits a side-confluent presentation $\langle X \mid R \rangle$ where X is a finite set. The left bound of $\langle X \mid R \rangle$ is a contracting homotopy for the Koszul complex of A if and only if $\langle X \mid R \rangle$ satisfy the reduction relations.*

In order the show that the extra-condition implies that the reduction relations hold, we have to describe it into algebraic relations. Recall that (ec) is:

$$\left(V^{\otimes m} \otimes \overline{R}\right) \cap \left(\overline{R} \otimes V^{\otimes m}\right) \subset V^{\otimes m-1} \otimes \overline{R} \otimes V, \text{ for every } 2 \leq m \leq N-1.$$

Applying the bijection $\theta^{-1}_{X^{(m+N)}}$ for every $2 \leq m \leq N-1$, the extra-condition is equivalent to:

$$S^{(m+N)}_{m-1} \preceq S^{(m+N)}_{m} \vee S^{(m+N)}_{0}, \text{ for every } 2 \leq m \leq N-1,$$

where $S^{(m+N)}_{i}$ is equal to $\theta^{-1}_{X^{(m+N)}}\left(V^{\otimes i} \otimes \overline{R} \otimes V^{\otimes m-i}\right)$. Thus, the extra-condition is equivalent to:

$$S^{(m+N)}_{m-1} \vee S^{(m+N)}_{m} \vee S^{(m+N)}_{0} = S^{(m+N)}_{m} \vee S^{(m+N)}_{0}, \text{ for every } 2 \leq m \leq N-1.$$

These relations are used to prove:

3.6. Proposition. *Let A be an N-homogeneous algebra. Assume that A admits an extra-confluent presentation $\langle X \mid R\rangle$. For every integers n and m such that $n \geq 1$ and $m \geq l_N(n+1)$, we have:*

$$F^{n,m}_1 \wedge \left(F^{n-1,m}_1 \vee F^{n-1,m}_2\right) = F^{n,m}_1 \wedge F^{n,m}_2.$$

We point the fact that the previous proposition is not true if the presentation is not side-confluent. Moreover, this proposition implies that the reduction relations hold. Thus, our main result is stated as follows:

3.7. Theorem. *Let A be an N-homogeneous algebra. If A admits an extra-confluent presentation $\langle X \mid R\rangle$, then the left bound of $\langle X \mid R\rangle$ is a contracting homotopy for the Koszul complex of A.*

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