DECENTRALIZED COLLABORATIVE LEARNING OF PERSONALIZED MODELS OVER NETWORKS

Aurélien Bellet (Inria MAGNET)

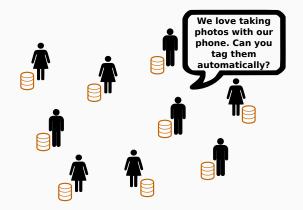
Joint work with Paul Vanhaesebrouck and Marc Tommasi

Distributed Optimization Workshop, Télécom ParisTech, 25/11/2016

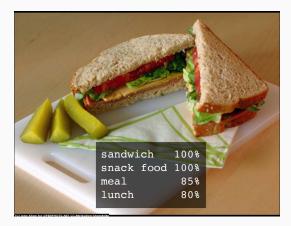
- 1. Broad context
- 2. Problem setting
- 3. Model propagation
- 4. Collaborative learning
- 5. Experiments
- 6. Future work

BROAD CONTEXT

LEARNING FROM PERSONAL DATA



LEARNING FROM PERSONAL DATA

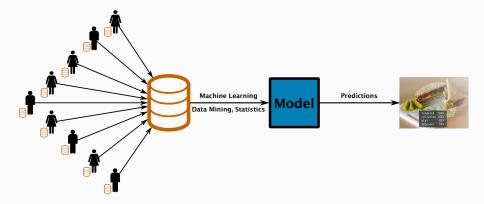


LEARNING FROM PERSONAL DATA



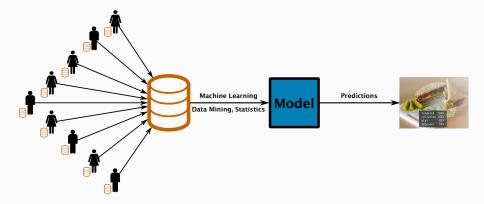
- Other examples of applications
 - Recommend content based on user activity logs
 - Predict health risks based on medical history

CURRENTLY DOMINANT APPROACH



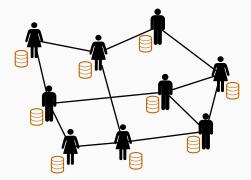
· Centralized data can be processed efficiently in a data center

CURRENTLY DOMINANT APPROACH



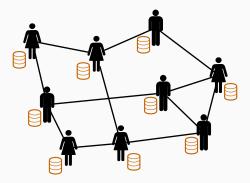
- · Lack of user control over its personal data
 - What is collected? Who can access it? How is it used and what for?
- Vulnerability to attacks / subpoenas
 - Yahoo data breach (500M users), Twitter / Wikileaks court orders
- Costly infrastructure for service provider

ALTERNATIVE : DECENTRALIZED ARCHITECTURE



- Personal data stays on user's device \rightarrow better control
- Peer-to-peer communications without a central server \rightarrow harder to collect data systematically (no single point of entry)

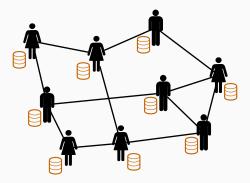
ALTERNATIVE : DECENTRALIZED ARCHITECTURE



Some scientific challenges

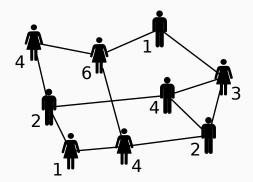
- 1. How to efficiently learn in a decentralized way under these communication constraints?
- 2. How to prevent malicious users from inferring sensitive data or manipulating the outcome to their advantage?

ALTERNATIVE : DECENTRALIZED ARCHITECTURE

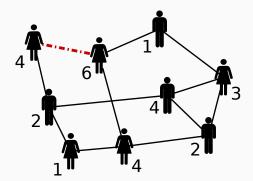


Some scientific challenges

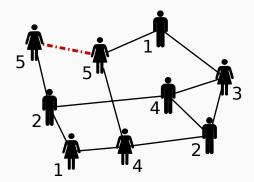
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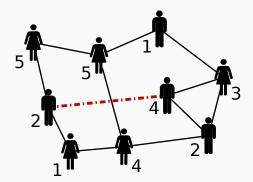
- Users wake up independently and asynchronously, select a random neighbor and exchange information
 - Equivalent view: at each step, activate a random network edge
- $\cdot\,$ Simple and asynchronous \rightarrow well suited to large networks



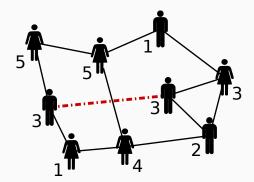
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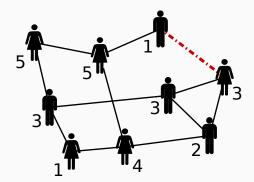
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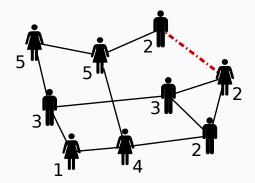
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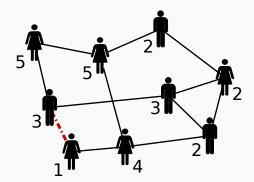
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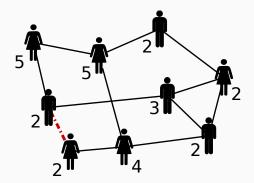
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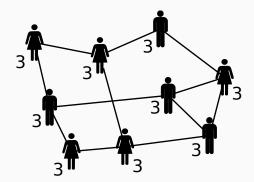
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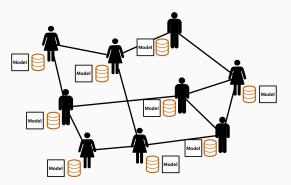
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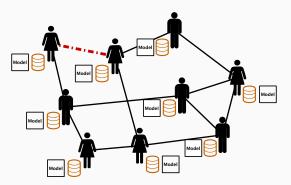
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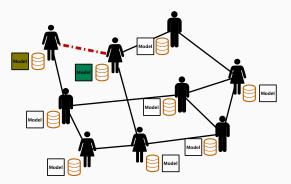
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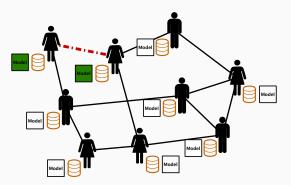
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 - 1. perform a local model update based on personal data
 - 2. average with neighbor



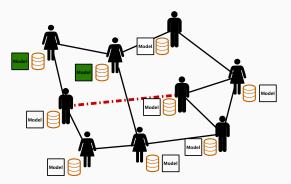
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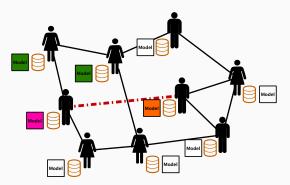
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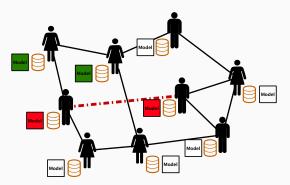
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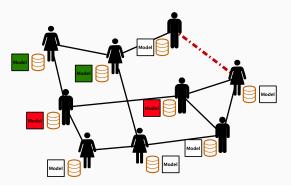
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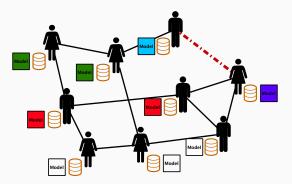
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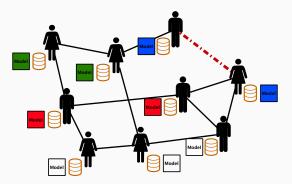
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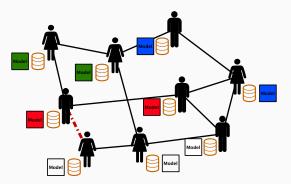
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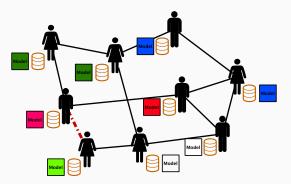
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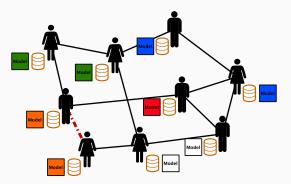
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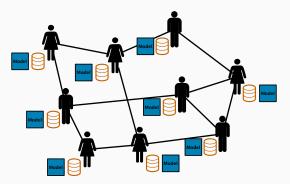
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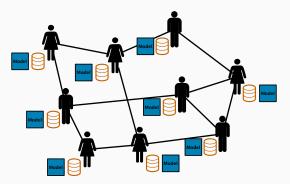
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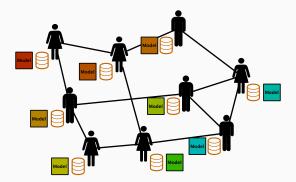
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THIS WORK: PERSONALIZED LEARNING

• Gossip algorithms to learn a personalized model for each user according to its own learning objective



• General idea: trade-off between model accuracy on local data and smoothness with respect to similar users

PROBLEM SETTING

PROBLEM SETTING

- A set $V = \llbracket n \rrbracket = \{1, \dots, n\}$ of *n* learning agents
- A convex loss function $\ell : \mathbb{R}^p \times \mathcal{X} \times \mathcal{Y}$
- Agent *i* has dataset $S_i = \{(x_i^j, y_i^j)\}_{j=1}^{m_i}$ of size $m_i \ge 0$ drawn i.i.d. from its own distribution μ_i over $\mathcal{X} \times \mathcal{Y}$
- Goal of agent *i*: learn a model $\theta_i \in \mathbb{R}^p$ with small expected loss

$$\mathbb{E}_{(x_i,y_i)\sim\mu_i}\ell(\theta_i;x_i,y_i)$$

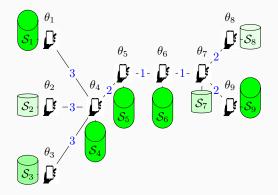
• In isolation, agent *i* can learn a "solitary" model

$$\theta_i^{sol} \in \operatorname*{arg\,min}_{\theta \in \mathbb{R}^p} \mathcal{L}_i(\theta) = \sum_{j=1}^{m_i} \ell(\theta; x_i^j, y_j^j)$$

• How to improve upon θ_i^{sol} with the help of other users?

- Network: weighted connected graph G = (V, E)
- $E \subseteq V \times V$ set of undirected edges
- Weight matrix $W \in \mathbb{R}^{n \times n}$: symmetric, nonnegative, with $W_{ij} = 0$ if $(i, j) \notin E$ or i = j
- **Simplifying assumption**: network weights are given and represent the underlying similarity between agents
 - Ex: movie recommendation task where the network is set up when users go to the same movie
- In a more general setup one would need to
 - estimate weights based on auxiliary observation or local data
 - construct a *k*-NN overlay on top of the physical communication network [Jelasity et al., 2009]

PROBLEM SETTING



- Agents have only a local view of the network
- They only know their neighborhood $N_i = \{j \neq i : W_{ij} > 0\}$ and the associated weights

MODEL PROPAGATION

MODEL PROPAGATION: PROBLEM FORMULATION

Main idea: smooth the solitary models over the network

- $c_i \in (0, 1]$: confidence in initial model θ_i^{sol}
 - Proportional to the number of training points *m_i*
- Find new set of models $\Theta \in \mathbb{R}^{n \times p}$ by solving

$$\min_{\Theta \in \mathbb{R}^{n \times p}} \mathcal{Q}_{MP}(\Theta) = \frac{1}{2} \left(\sum_{i < j}^{n} W_{ij} \| \theta_i - \theta_j \|^2 + \mu \sum_{i=1}^{n} D_{ii} c_i \| \theta_i - \theta_i^{sol} \|^2 \right)$$

- Trade-off between smoothing models within neighborhoods and not diverging too much from confident models
- Term $D_{ii} = \sum_{i} W_{ij}$ is just for normalization
- Strict generalization of Label Propagation (LP) [Zhou et al., 2004]
 - Constant c_i 's \rightarrow recover LP
 - Variable c_i 's \rightarrow cannot be expressed as LP
 - Our gossip algorithm will readily apply to LP!

- · Cannot use closed-form solution (requires global knowledge)
- \cdot The following iteration converges to the same quantity

$$\Theta(t+1) = (\alpha l + \bar{\alpha}C)^{-1} \left(\alpha P \Theta(t) + \bar{\alpha}C \Theta^{sol} \right)$$

- $P = D^{-1}W$ (stochastic similarity matrix)
- + $\alpha \in (0, 1)$ such that $\mu = (1 \alpha)/\alpha$, $\bar{\alpha} = 1 \alpha$
- Decomposes into

$$\theta_i(t+1) = \frac{1}{\alpha + \bar{\alpha}c_i} \left(\alpha \sum_{j \in \mathcal{N}_i} \frac{W_{ij}}{D_{ii}} \theta_j(t) + \bar{\alpha}c_i \theta_i^{sol} \right)$$

- This is a decentralized but synchronous process
 - Assumes availability of global clock
 - · Synchronization incurs delays (must wait for everyone to finish)
 - All neighbors must be contacted at each step

- Each agent has a local Poisson clock and wakes up when it ticks \rightarrow equivalent to activating a random node at each step t
- Idea of our algorithm: each agent *i* maintains a (possibly outdated) knowledge $\widetilde{\Theta}_i(t) \in \mathbb{R}^{n \times p}$ of its neighbors' models
 - $\widetilde{\Theta}_{i}^{i}(t) \in \mathbb{R}^{p}$: agent *i*'s model at time *t*
 - for $j \neq i$, $\widetilde{\Theta}_{i}^{j}(t) \in \mathbb{R}^{p}$: agent *i*'s last knowledge of the model of *j*
 - For $j \notin \mathcal{N}_i \cup \{i\}$ and any t > 0, we maintain $\widetilde{\Theta}_i^j(t) = 0$

- At step *t*, some agent *i* wakes up and two actions are performed
 - 1. Communication step: agent *i* selects a random neighbor $j \in N_i$ w.p. π_i^j and both agents update their knowledge of each other:

$$\widetilde{\Theta}^{j}_{i}(t+1) = \widetilde{\Theta}^{j}_{j}(t)$$
 and $\widetilde{\Theta}^{j}_{j}(t+1) = \widetilde{\Theta}^{j}_{i}(t)$,

2. Update step: agents *i* and *j* update their own models based on current knowledge. For $l \in \{i, j\}$:

$$\widetilde{\Theta}_{l}^{l}(t+1) = (\alpha + \bar{\alpha}c_{l})^{-1} \Big(\alpha \sum_{k \in \mathcal{N}_{l}} \frac{W_{lk}}{D_{ll}} \widetilde{\Theta}_{l}^{k}(t+1) + \bar{\alpha}c_{l}\theta_{l}^{sol} \Big).$$

- All other variables in the network remain unchanged
- For any $i \in [n]$, $\pi_i \in [0, 1]^n$ must satisfy $\sum_{j=1}^n \pi_i^j = 1$ and $\pi_i^j > 0$ if and only if $j \in \mathcal{N}_i$

Theorem ([Vanhaesebrouck et al., 2016])

Let $\widetilde{\Theta}(0) \in \mathbb{R}^{n^2 \times p}$ be some initial value and $(\widetilde{\Theta}(t))_{t \in \mathbb{N}}$ be the sequence generated by our algorithm. Let $\Theta^* = \arg\min_{\Theta \in \mathbb{R}^{n \times p}} \mathcal{Q}_{MP}(\Theta)$ be the optimal solution to model propagation. For any $i \in [n]$,

$$\lim_{t\to\infty}\mathbb{E}\left[\widetilde{\Theta}_{i}^{j}(t)\right]=\Theta_{j}^{\star} \text{ for } j\in\mathcal{N}_{i}\cup\{i\}.$$

Sketch of proof

• Rewrite algorithm as a random iterative process over $\widetilde{\Theta} \in \mathbb{R}^{n^2 \times p}$:

$$\widetilde{\Theta}(t+1) = A(t)\widetilde{\Theta}(t) + b(t)$$

- Show that spectral radius of $\mathbb{E}[A(t)]$ is smaller than 1
- Exhibit convergence to desired quantity

COLLABORATIVE LEARNING

- Model propagation is very simple but forgets data
- · Alternative: learn / propagate models simultaneously by solving

$$\min_{\Theta \in \mathbb{R}^{n \times p}} \mathcal{Q}_{CL}(\Theta) = \sum_{i < j}^{n} W_{ij} \|\theta_i - \theta_j\|^2 + \mu \sum_{i=1}^{n} D_{ii} \mathcal{L}_i(\theta_i)$$

- Trade-off between smoothing models within neighborhoods and good accuracy on local datasets
- Note: confidence is built in second term
- More flexibility in settings where different parameter values may lead to similar predictions

• We will rely on ADMM [Boyd et al., 2011, Wei and Ozdaglar, 2012], which is popular for solving decentralized consensus problems

$$\min_{\theta \in \mathbb{R}^p} \sum_{i=1}^n \mathcal{L}_i(\theta)$$

- Main idea: reformulate our problem as a partial consensus and decouple the objectives
- Let Θ_i be the set of $|\mathcal{N}_i| + 1$ variables $\theta_j \in \mathbb{R}^p$ for $j \in \mathcal{N}_i \cup \{i\}$, denote θ_j by Θ_j^i and define

$$\mathcal{Q}_{CL}^{i}(\Theta_{i}) = \frac{1}{2} \sum_{j \in \mathcal{N}_{i}} W_{ij} \|\Theta_{i}^{i} - \Theta_{j}^{j}\|^{2} + \mu D_{ii} \mathcal{L}_{i}(\Theta_{i}^{i}),$$

• We can rewrite our problem as $\min_{\Theta \in \mathbb{R}^{n \times p}} \sum_{i=1}^{n} Q_{CL}^{i}(\Theta_{i})$

- *Decoupling*: introduce a local copy $\widetilde{\Theta}_i \in \mathbb{R}^{(|\mathcal{N}_i|+1) \times p}$ of the decision variables Θ_i for each agent *i*
- Partial consensus: impose equality constraints on the variables $\widetilde{\Theta}_i^i = \widetilde{\Theta}_j^i$ for all $i \in [\![n]\!], j \in \mathcal{N}_i$
 - $\cdot\,$ two neighboring agents agree on each other's personalized model
- Further introduce 4 secondary variables $Z_{ei}^i, Z_{ej}^i, Z_{ei}^j$ and Z_{ej}^j for each edge e = (i, j)
 - can be viewed as estimates of the models $\widetilde{\Theta}_i$ and $\widetilde{\Theta}_j$ known by each end of e
 - will allow efficient decomposition of ADMM updates

REFORMULATION AS PARTIAL CONSENSUS PROBLEM

• Final reformulation: denoting $\widetilde{\Theta} = [\widetilde{\Theta}_1^{\top}, \dots, \widetilde{\Theta}_n^{\top}]^{\top} \in \mathbb{R}^{(2|E|+n) \times p}$ and $Z \in \mathbb{R}^{4|E| \times p}$

$$\begin{array}{ll} \min_{\widetilde{\Theta} \in \mathbb{R}^{(2|\mathcal{E}|+n) \times p} \\ Z \in \mathcal{C}_{\mathcal{E}} \end{array}} & \sum_{i=1}^{n} \mathcal{Q}_{CL}^{i}(\widetilde{\Theta}_{i}) \\ \text{s.t. } \forall e = (i,j) \in E, \quad \begin{cases} Z_{ei}^{i} = \widetilde{\Theta}_{i}^{i}, \ Z_{ei}^{j} = \widetilde{\Theta}_{i}^{j} \\ Z_{ej}^{i} = \widetilde{\Theta}_{j}^{i}, \ Z_{ej}^{i} = \widetilde{\Theta}_{j}^{i} \end{cases}$$

where $C_E = \{Z \in \mathbb{R}^{4|E| \times p} \mid Z_{ei}^i = Z_{ej}^i, Z_{ej}^j = Z_{ei}^j \text{ for all } e = (i, j) \in E\}$

- Constraints involving $\widetilde{\Theta}$ can be written $D\widetilde{\Theta} + HZ = 0$ where
 - H = -I of dimension $4|E| \times 4|E|$ is diagonal invertible
 - · $D \in \mathbb{R}^{4|E| \times (2|E|+n)}$ contains exactly one entry of 1 in each row
- Assumptions of [Wei and Ozdaglar, 2013] satisfied

• The augmented Lagrangian of the problem is

$$L_{\rho}(\widetilde{\Theta}, Z, \Lambda) = \sum_{i=1}^{n} L_{\rho}^{i}(\widetilde{\Theta}_{i}, Z_{i}, \Lambda_{i}),$$

where $\rho > 0$ is a penalty parameter, $Z \in C_E$ and

$$\begin{split} L^{i}_{\rho}(\widetilde{\Theta}_{i}, Z_{i}, \Lambda_{i}) &= \mathcal{Q}^{i}_{CL}(\widetilde{\Theta}_{i}) + \sum_{j:e=(i,j)\in E} \left[\Lambda^{i}_{ei}(\widetilde{\Theta}^{j}_{i} - Z^{i}_{ei}) \right. \\ &+ \Lambda^{j}_{ei}(\widetilde{\Theta}^{j}_{i} - Z^{j}_{ei}) + \frac{\rho}{2} \left(\left\| \widetilde{\Theta}^{j}_{i} - Z^{j}_{ei} \right\|^{2} + \left\| \widetilde{\Theta}^{j}_{i} - Z^{j}_{ei} \right\|^{2} \right) \right]. \end{split}$$

- ADMM iteratively minimize the augmented Lagrangian by alternating
 - 1. minimization w.r.t. primal variable $\widetilde{\Theta}$
 - 2. minimization w.r.t. secondary variable Z
 - 3. update of the dual variable Λ

- Assume that agent *i* wakes up at step *t* and selects $j \in N_i$. Denoting e = (i, j)
 - 1. Agent *i* updates its primal variables:

$$\widetilde{\Theta}_i(t+1) = \arg\min_{\Theta \in \mathbb{R}^{(|\mathcal{N}_i|+1) \times p}} L^i_{\rho}(\Theta, Z_i(t), \Lambda_i(t)),$$

and sends $\widetilde{\Theta}_{i}^{i}(t + 1), \widetilde{\Theta}_{i}^{j}(t + 1), \Lambda_{ei}^{i}(t), \Lambda_{ei}^{j}(t)$ to agent *j*. Agent *j* executes the same steps w.r.t. *i*.

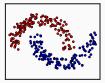
- 2. Using $\widetilde{\Theta}_{j}^{i}(t+1)$, $\widetilde{\Theta}_{i}^{i}(t+1)$, $\Lambda_{ej}^{i}(t)$, $\Lambda_{ej}^{i}(t)$ received from *j*, agent *i* updates its secondary variables $Z_{ei}^{i}(t+1)$ and $Z_{ei}^{j}(t+1)$ (closed form). Agent *j* updates its secondary variables symmetrically
- 3. Agent *i* updates its dual variables $\Lambda_{ei}^{i}(t + 1)$ and $\Lambda_{ei}^{j}(t + 1)$ (closed-form). Agent *j* updates its dual variables symmetrically.
- Convergence in O(1/t) [Wei and Ozdaglar, 2013]

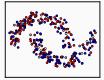
EXPERIMENTS

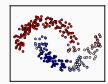
- We consider n = 300 agents and a 1D mean estimation task $(\log f(\theta; x_i) = ||\theta x_i||^2)$
 - Network topology derived from the two moons dataset
 - Each agent *i* has a true 1D Gaussian distribution μ_i centered at -1 or +1 depending on the moon it belongs to
 - Each agent *i* receives a random number m_i of samples from μ_i

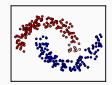
COLLABORATIVE MEAN ESTIMATION

- Confidence values help a lot for imbalanced datasets
- Our MP algorithm has fast convergence without synchronization







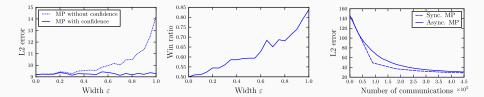


(a) Ground models

(b) Solitary models

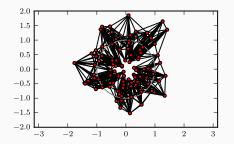
(c) MP without confidence

(d) MP with confidence



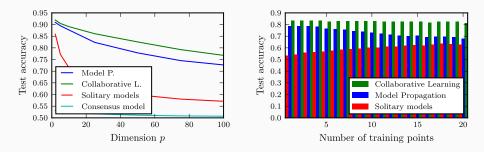
COLLABORATIVE LINEAR CLASSIFICATION

- We consider a set of n = 100 agents and a linear classification task in \mathbb{R}^p (with hinge loss)
 - Target models lie in a 2D subspace, network weights based on the angle between true models
 - Each agent *i* receives a random number *m_i* of samples with label given by the prediction of target model (plus noise)



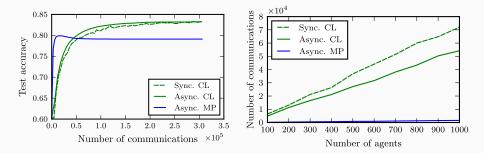
COLLABORATIVE LINEAR CLASSIFICATION

- Both CL and MP provide great improvements over local models
- CL consistently outperforms MP by significant margin
- Effectively compensating for training size imbalance



COLLABORATIVE LINEAR CLASSIFICATION

- CL algorithm converges as fast as a synchronous approach
- MP much faster to converge and can be used as warm-start to speed up CL
- Number of iterations to converge to near-optimal accuracy scales linearly with network size



FUTURE WORK

FUTURE WORK (AND ADS)

- Study link between similarity graph and generalization performance
- · Generic methods to estimate/learn graph weights
- Decentralized discovery of similar peers
- Privacy-preserving mechanisms

Quick ads

- Make sure you attend the NIPS 2016 workshop on Private Multi-Party Machine Learning!
- We have **many open positions** in our Inria team (tenured, postdocs, PhDs, Master internships) with exciting projects!

THANK YOU FOR YOUR ATTENTION! QUESTIONS?

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