Error Propagation for Approximate Policy and Value Iteration

Amir massoud Farahmand
Department of Computing Science
University of Alberta
Edmonton, Canada, T6G 2E8
`amirf@ualberta.ca`

Rémi Munos
Sequel Project, INRIA Lille
Lille, France
`remi.munos@inria.fr`

Csaba Szepesvári *
Department of Computing Science
University of Alberta
Edmonton, Canada, T6G 2E8
`szepesva@ualberta.ca`

Abstract

We address the question of how the approximation error/Bellman residual at each iteration of the Approximate Policy/Value Iteration algorithms influences the quality of the resulted policy. We quantify the performance loss as the L_p norm of the approximation error/Bellman residual at each iteration. Moreover, we show that the performance loss depends on the expectation of the squared Radon-Nikodym derivative of a certain distribution rather than its supremum – as opposed to what has been suggested by the previous results. Also our results indicate that the contribution of the approximation/Bellman error to the performance loss is more prominent in the later iterations of API/AVI, and the effect of an error term in the earlier iterations decays exponentially fast.

1 Introduction

The exact solution for the reinforcement learning (RL) and planning problems with large state space is difficult or impossible to obtain, so one usually has to aim for approximate solutions. Approximate Policy Iteration (API) and Approximate Value Iteration (AVI) are two classes of iterative algorithms to solve RL/Planning problems with large state spaces. They try to approximately find the fixed-point solution of the Bellman optimality operator.

AVI starts from an initial value function V_0 (or Q_0), and iteratively applies an *approximation* of T^*, the Bellman optimality operator, (or T^π for the policy evaluation problem) to the previous estimate, i.e., $V_{k+1} \approx T^* V_k$. In general, V_{k+1} is not equal to $T^* V_k$ because (1) we do not have direct access to the Bellman operator but only some samples from it, and (2) the function space in which V belongs is not representative enough. Thus there would be an approximation error $\varepsilon_k = T^* V_k - V_{k+1}$ between the result of the exact VI and AVI.

Some examples of AVI-based approaches are tree-based Fitted Q-Iteration of Ernst et al. [1], multi-layer perceptron-based Fitted Q-Iteration of Riedmiller [2], and regularized Fitted Q-Iteration of Farahmand et al. [3]. See the work of Munos and Szepesvári [4] for more information about AVI.

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API is another iterative algorithm to find an approximate solution to the fixed point of the Bellman *optimality* operator. It starts from a policy π_0, and then approximately *evaluates* that policy π_0, i.e., it finds a Q_0 that satisfies $T^{\pi_0} Q_0 \approx Q_0$. Afterwards, it performs a policy improvement step, which is to calculate the greedy policy with respect to (w.r.t.) the most recent action-value function, to get a new policy π_1, i.e., $\pi_1(\cdot) = \arg\max_{a \in \mathcal{A}} Q_0(\cdot, a)$. The policy iteration algorithm continues by approximately evaluating the newly obtained policy π_1 to get Q_1 and repeating the whole process again, generating a sequence of policies and their corresponding approximate action-value functions $Q_0 \to \pi_1 \to Q_1 \to \pi_2 \to \cdots$. Same as AVI, we may encounter a difference between the approximate solution Q_k ($T^{\pi_k} Q_k \approx Q_k$) and the true value of the policy Q^{π_k}, which is the solution of the fixed-point equation $T^{\pi_k} Q^{\pi_k} = Q^{\pi_k}$. Two convenient ways to describe this error is either by the Bellman residual of Q_k ($\varepsilon_k = Q_k - T^{\pi_k} Q_k$) or the policy evaluation approximation error ($\varepsilon_k = Q_k - Q^{\pi_k}$).

API is a popular approach in RL literature. One well-known algorithm is LSPI of Lagoudakis and Parr [5] that combines Least-Squares Temporal Difference (LSTD) algorithm (Bradtke and Barto [6]) with a policy improvement step. Another API method is to use the Bellman Residual Minimization (BRM) and its variants for policy evaluation and iteratively apply the policy improvement step (Antos et al. [7], Maillard et al. [8]). Both LSPI and BRM have many extensions: Farahmand et al. [9] introduced a nonparametric extension of LSPI and BRM and formulated them as an optimization problem in a reproducing kernel Hilbert space and analyzed its statistical behavior. Kolter and Ng [10] formulated an l_1 regularization extension of LSTD. See Xu et al. [11] and Jung and Polani [12] for other examples of kernel-based extension of LSTD/LSPI, and Taylor and Parr [13] for a unified framework. Also see the proto-value function-based approach of Mahadevan and Maggioni [14] and iLSTD of Geramifard et al. [15].

A crucial question in the applicability of API/AVI, which is the main topic of this work, is to understand how either the approximation error or the Bellman residual at each iteration of API or AVI affects the quality of the resulted policy. Suppose we run API/AVI for K iterations to obtain a policy π_K. Does the knowledge that all ε_ks are small (maybe because we have had a lot of samples and used powerful function approximators) imply that V^{π_K} is close to the optimal value function V^* too? If so, how does the errors occurred at a certain iteration k **propagate** through iterations of API/AVI and affect the final performance loss?

There have already been some results that partially address this question. As an example, Proposition 6.2 of Bertsekas and Tsitsiklis [16] shows that for API applied to a finite MDP, we have $\limsup_{k \to \infty} \|V^* - V^{\pi_k}\|_\infty \leq \frac{2\gamma}{(1-\gamma)^2} \limsup_{k \to \infty} \|V^{\pi_k} - V_k\|_\infty$ where γ is the discount facto. Similarly for AVI, if the approximation errors are uniformly bounded ($\|T^* V_k - V_{k+1}\|_\infty \leq \varepsilon$), we have $\limsup_{k \to \infty} \|V^* - V^{\pi_k}\|_\infty \leq \frac{2\gamma}{(1-\gamma)^2} \varepsilon$ (Munos [17]).

Nevertheless, most of these results are pessimistic in several ways. One reason is that they are expressed as the supremum norm of the approximation errors $\|V^{\pi_k} - V_k\|_\infty$ or the Bellman error $\|Q_k - T^{\pi_k} Q_k\|_\infty$. Compared to L_p norms, the supremum norm is conservative. It is quite possible that the result of a learning algorithm has a small L_p norm but a very large L_∞ norm. Therefore, it is desirable to have a result expressed in L_p norm of the approximation/Bellman residual ε_k.

In the past couple of years, there have been attempts to extend L_∞ norm results to L_p ones [18, 17, 7]. As a typical example, we quote the following from Antos et al. [7]:

Proposition 1 (Error Propagation for API – [7]). *Let $p \geq 1$ be a real and K be a positive integer. Then, for any sequence of functions $\{Q^{(k)}\} \subset B(\mathcal{X} \times \mathcal{A}; Q_{max})(0 \leq k < K)$, the space of Q_{max}-bounded measurable functions, and their corresponding Bellman residuals $\varepsilon_k = Q_k - T^\pi Q_k$, the following inequalities hold:*

$$\|Q^* - Q^{\pi_K}\|_{p,\rho} \leq \frac{2\gamma}{(1-\gamma)^2} \left(C_{\rho,\nu}^{1/p} \max_{0 \leq k < K} \|\varepsilon_k\|_{p,\nu} + \gamma^{\frac{K}{p} - 1} R_{max} \right),$$

where R_{max} is an upper bound on the magnitude of the expected reward function and

$$C_{\rho,\nu} = (1-\gamma)^2 \sum_{m \geq 1} m \gamma^{m-1} \sup_{\pi_1, \ldots, \pi_m} \left\| \frac{d\left(\rho P^{\pi_1} \cdots P^{\pi_m}\right)}{d\nu} \right\|_\infty.$$

This result indeed uses L_p norm of the Bellman residuals and is an improvement over results like Bertsekas and Tsitsiklis [16, Proposition 6.2], but still is pessimistic in some ways and does

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not answer several important questions. For instance, this result implies that the uniform-over-all-iterations upper bound $\max_{0 \leq k < K} \|\varepsilon_k\|_{p,\nu}$ is the quantity that determines the performance loss. One may wonder if this condition is really necessary, and ask whether it is better to put more emphasis on earlier/later iterations? Or another question is whether the appearance of terms in the form of $\|\frac{d(\rho P^{\pi_1} \ldots P^{\pi_m})}{d\nu}\|_\infty$ is intrinsic to the difficulty of the problem or can be relaxed.

The goal of this work is to answer these questions and to provide tighter upper bounds on the performance loss of API/AVI algorithms. These bounds help one understand what factors contribute to the difficulty of a learning problem. We base our analysis on the work of Munos [17], Antos et al. [7], Munos [18] and provide upper bounds on the performance loss in the form of $\|V^* - V^{\pi_k}\|_{1,\rho}$ (the expected loss weighted according to the evaluation probability distribution ρ – this is defined in Section 2) for API (Section 3) and AVI (Section 4). This performance loss depends on a certain function of ν-weighted L_2 norms of ε_ks, in which ν is the data sampling distribution, and $C_{\rho,\nu}(K)$ that depends on the MDP, two probability distributions ρ and ν, and the number of iterations K.

In addition to relating the performance loss to L_p norm of the Bellman residual/approximation error, this work has three main contributions that to our knowledge have not been considered before: (1) We show that the performance loss depends on the **expectation** of the squared Radon-Nikodym derivative of a certain distribution, to be specified in Section 3, rather than its supremum. The difference between this expectation and the supremum can be considerable. For instance, for a finite state space with N states, the ratio can be of order $O(N^{1/2})$. (2) The contribution of the Bellman/approximation error to the performance loss is more prominent in later iterations of API/AVI. and the effect of an error term in early iterations decays exponentially fast. (3) There are certain structures in the definition of concentrability coefficients that have not been explored before. We thoroughly discuss these **qualitative/structural** improvements in Section 5.

2   Background

In this section, we provide a very brief summary of some of the concepts and definitions from the theory of Markov Decision Processes (MDP) and reinforcement learning (RL) and a few other notations. For further information about MDPs and RL the reader is referred to [19, 16, 20, 21].

A *finite-action discounted MDP* is a 5-tuple $(\mathcal{X}, \mathcal{A}, P, \mathcal{R}, \gamma)$, where $\mathcal{X}$ is a measurable state space, $\mathcal{A}$ is a finite set of actions, P is the probability transition kernel, $\mathcal{R}$ is the reward kernel, and $0 \leq \gamma < 1$ is the discount factor. The transition kernel P is a mapping with domain $\mathcal{X} \times \mathcal{A}$ evaluated at $(x, a) \in \mathcal{X} \times \mathcal{A}$ that gives a distribution over $\mathcal{X}$, which we shall denote by $P(\cdot|x, a)$. Likewise, $\mathcal{R}$ is a mapping with domain $\mathcal{X} \times \mathcal{A}$ that gives a distribution of immediate reward over $\mathbb{R}$, which is denoted by $\mathcal{R}(\cdot|x, a)$. We denote $r(x, a) = \mathbb{E}[\mathcal{R}(\cdot|x, a)]$, and assume that its absolute value is bounded by $R_{\max}$.

A mapping $\pi : \mathcal{X} \to \mathcal{A}$ is called a deterministic Markov stationary policy, or just a *policy* in short. Following a policy π in an MDP means that at each time step $A_t = \pi(X_t)$. Upon taking action A_t at X_t, we receive reward $R_t \sim \mathcal{R}(\cdot|x, a)$, and the Markov chain evolves according to $X_{t+1} \sim P(\cdot|X_t, A_t)$. We denote the probability transition kernel of following a policy π by P^π, i.e., $P^\pi(dy|x) = P(dy|x, \pi(x))$.

The value function V^π for a policy π is defined as $V^\pi(x) \triangleq \mathbb{E}\left[\sum_{t=0}^\infty \gamma^t R_t \,\middle|\, X_0 = x\right]$ and the action-value function is defined as $Q^\pi(x, a) \triangleq \mathbb{E}\left[\sum_{t=0}^\infty \gamma^t R_t \,\middle|\, X_0 = x, A_0 = a\right]$. For a discounted MDP, we define the *optimal value* and *action-value* functions by $V^*(x) = \sup_\pi V^\pi(x)$ $(\forall x \in \mathcal{X})$ and $Q^*(x, a) = \sup_\pi Q^\pi(x, a)$ $(\forall x \in \mathcal{X}, \forall a \in \mathcal{A})$. We say that a policy π^* is *optimal* if it achieves the best values in every state, i.e., if $V^{\pi^*} = V^*$. We say that a policy π is *greedy* w.r.t. an action-value function Q and write $\pi = \hat{\pi}(\cdot; Q)$, if $\pi(x) \in \arg\max_{a \in \mathcal{A}} Q(x, a)$ holds for all $x \in \mathcal{X}$. Similarly, the policy π is greedy w.r.t. V, if for all $x \in \mathcal{X}$, $\pi(x) \in \arg\max_{a \in \mathcal{A}} \int P(dx'|x, a)[r(x, a) + \gamma V(x')]$ (If there exist multiple maximizers, some maximizer is chosen in an arbitrary deterministic manner). Greedy policies are important because a greedy policy w.r.t. Q^* (or V^*) is an optimal policy. Hence, knowing Q^* is sufficient for behaving optimally (cf. Proposition 4.3 of [19]).

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We define the Bellman operator for a policy π as $(T^\pi V)(x) \triangleq r(x, \pi(x)) + \gamma \int V^\pi(x')P(dx'|x, a)$ and $(T^\pi Q)(x, a) \triangleq r(x, a) + \gamma \int Q(x', \pi(x'))P(dx'|x, a)$. Similarly, the Bellman optimality operator is defined as $(T^*V)(x) \triangleq \max_a \left\{ r(x, a) + \gamma \int V(x')P(dx'|x, a) \right\}$ and $(T^*Q)(x, a) \triangleq r(x, a) + \gamma \int \max_{a'} Q(x', a')P(dx'|x, a)$.

For a measurable space $\mathcal{X}$, with a σ-algebra $\sigma_\mathcal{X}$, we define $\mathcal{M}(\mathcal{X})$ as the set of all probability measures over $\sigma_\mathcal{X}$. For a probability measure $\rho \in \mathcal{M}(\mathcal{X})$ and the transition kernel P^π, we define $\rho P^\pi(dx') = \int P(dx'|x, \pi(x))d\rho(x)$. In words, $\rho(P^\pi)^m \in \mathcal{M}(\mathcal{X})$ is an m-step-ahead probability distribution of states if the starting state distribution is ρ and we follow P^π for m steps. In what follows we shall use $\|V\|_{p,\nu}$ to denote the $L^p(\nu)$-norm of a measurable function $V : \mathcal{X} \to \mathbb{R}$: $\|V\|_{p,\nu}^p \triangleq \nu|V|^p \triangleq \int_\mathcal{X} |V(x)|^p d\nu(x)$. For a function $Q : \mathcal{X} \times \mathcal{A} \mapsto \mathbb{R}$, we define $\|Q\|_{p,\nu}^p \triangleq \frac{1}{|\mathcal{A}|} \sum_{a \in \mathcal{A}} \int_\mathcal{X} |Q(x, a)|^p d\nu(x)$.

3 Approximate Policy Iteration

Consider the API procedure and the sequence $Q_0 \to \pi_1 \to Q_1 \to \pi_2 \to \cdots \to Q_{K-1} \to \pi_K$, where π_k is the greedy policy w.r.t. Q_{k-1} and Q_k is the approximate action-value function for policy π_k. For the sequence $\{Q_k\}_{k=0}^{K-1}$, denote the **B**ellman **R**esidual (BR) and policy **A**pproximation **E**rror (AE) at each iteration by

$$\varepsilon_k^{\text{BR}} = Q_k - T^{\pi_k}Q_k, \tag{1}$$

$$\varepsilon_k^{\text{AE}} = Q_k - Q^{\pi_k}. \tag{2}$$

The goal of this section is to study the effect of ν-weighted L_{2p} norm of the Bellman residual sequence $\{\varepsilon_k^{\text{BR}}\}_{k=0}^{K-1}$ or the policy evaluation approximation error sequence $\{\varepsilon_k^{\text{AE}}\}_{k=0}^{K-1}$ on the performance loss $\|Q^* - Q^{\pi_K}\|_{p,\rho}$ of the outcome policy π_K.

The choice of ρ and ν is arbitrary, however, a natural choice for ν is the sampling distribution of the data, which is used by the policy evaluation module. On the other hand, the probability distribution ρ reflects the importance of various regions of the state space and is selected by the practitioner. One common choice, though not necessarily the best, is the stationary distribution of the optimal policy.

Because of the dynamical nature of MDP, the performance loss $\|Q^* - Q^{\pi_K}\|_{p,\rho}$ depends on the difference between the sampling distribution ν and the future-state distribution in the form of $\rho P^{\pi_1} P^{\pi_2} \cdots$. The precise form of this dependence will be formalized in Theorems 3 and 4. Before stating the results, we require to define the following *concentrability* coefficients.

Definition 2 (Expected Concentrability of the Future-State Distribution). *Given $\rho, \nu \in \mathcal{M}(\mathcal{X})$, $\nu \ll \lambda$[1] (λ is the Lebesgue measure), $m \geq 0$, and an arbitrary sequence of stationary policies $\{\pi_m\}_{m \geq 1}$, let $\rho P^{\pi_1} P^{\pi_2} \ldots P^{\pi_m} \in \mathcal{M}(\mathcal{X})$ denote the future-state distribution obtained when the first state is distributed according to ρ and then we follow the sequence of policies $\{\pi_k\}_{k=1}^m$.*

Define the following concentrability coefficients that is used in API analysis:

$$c_{PI_1,\rho,\nu}(m_1, m_2; \pi) \triangleq \left(\mathbb{E}_{X \sim \nu} \left[\left| \frac{d\left(\rho(P^{\pi^*})^{m_1}(P^\pi)^{m_2}\right)}{d\nu}(X) \right|^2 \right] \right)^{\frac{1}{2}},$$

$$c_{PI_2,\rho,\nu}(m_1, m_2; \pi_1, \pi_2) \triangleq \left(\mathbb{E}_{X \sim \nu} \left[\left| \frac{d\left(\rho(P^{\pi^*})^{m_1}(P^{\pi_1})^{m_2}P^{\pi_2}\right)}{d\nu}(X) \right|^2 \right] \right)^{\frac{1}{2}},$$

$$c_{PI_3,\rho,\nu} \triangleq \left(\mathbb{E}_{X \sim \nu} \left[\left| \frac{d\left(\rho P^{\pi^*}\right)}{d\nu}(X) \right|^2 \right] \right)^{\frac{1}{2}},$$

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[1]For two measures ν_1 and ν_2 on the same measurable space, we say that ν_1 is absolutely continuous with respect to ν_2 (or ν_2 dominates ν_1) and denote $\nu_1 \ll \nu_2$ iff $\nu_2(A) = 0 \Rightarrow \nu_1(A) = 0$.

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with the understanding that if the future-state distribution $\rho(P^{\pi^*})^{m_1}(P^\pi)^{m_2}$ (or $\rho(P^{\pi^*})^{m_1}(P^{\pi_1})^{m_2}P^{\pi_2}$ or ρP^{π^*}) is not absolutely continuous w.r.t. ν, then we take $c_{PI_1,\rho,\nu}(m_1,m_2;\pi) = \infty$ (similar for others).

Also define the following concentrability coefficient that is used in AVI analysis:

$$c_{VI,\rho,\nu}(m_1,m_2;\pi) \triangleq \left(\mathbb{E}_{X\sim\nu}\left[\left| \frac{d\left(\rho(P^\pi)^{m_1}(P^{\pi^*})^{m_2}\right)}{d\nu}(X) \right|^2 \right] \right)^{\frac{1}{2}},$$

with the understanding that if the future-state distribution $\rho(P^{\pi^*})^{m_1}(P^\pi)^{m_2}$ is not absolutely continuous w.r.t. ν, then we take $c_{VI,\rho,\nu}(m_1,m_2;\pi) = \infty$.

In order to compactly present our results, we define the following notation:

$$\alpha_k = \begin{cases} \frac{(1-\gamma)\gamma^{K-k-1}}{1-\gamma^{K+1}} & 0 \leq k < K, \\ \frac{(1-\gamma)\gamma^K}{1-\gamma^{K+1}} & k = K. \end{cases} \tag{3}$$

Theorem 3 (Error Propagation for API). *Let $p \geq 1$ be a real number, K be a positive integer, and $Q_{max} \leq \frac{R_{max}}{1-\gamma}$. Then for any sequence $\{Q_k\}_{k=0}^{K-1} \subset B(\mathcal{X} \times \mathcal{A}, Q_{max})$ (space of Q_{max}-bounded measurable functions defined on $\mathcal{X} \times \mathcal{A}$) and the corresponding sequence $\{\varepsilon_k\}_{k=0}^{K-1}$ defined in (1) or (2), we have*

$$\|Q^* - Q^{\pi_K}\|_{p,\rho} \leq \frac{2\gamma}{(1-\gamma)^2} \left[\inf_{r\in[0,1]} C_{PI(BR/AE),\rho,\nu}^{\frac{1}{2p}}(K;r) \mathcal{E}^{\frac{1}{2p}}(\varepsilon_0,\ldots,\varepsilon_{K-1};r) + \gamma^{\frac{K}{p}-1} R_{max} \right].$$

where $\mathcal{E}(\varepsilon_0,\ldots,\varepsilon_{K-1};r) = \sum_{k=0}^{K-1} \alpha_k^{2r} \|\varepsilon_k\|_{2p,\nu}^{2p}$.

(a) If $\varepsilon_k = \varepsilon^{BR}$ for all $0 \leq k < K$, we have

$$C_{PI(BR),\rho,\nu}(K;r) = \left(\frac{1-\gamma}{2}\right)^2 \sup_{\pi'_0,\ldots,\pi'_K} \sum_{k=0}^{K-1} \alpha_k^{2(1-r)} \left(\sum_{m\geq 0} \gamma^m \Big(c_{PI_1,\rho,\nu}(K-k-1,m+1;\pi'_{k+1}) + \right.$$

$$\left. c_{PI_1,\rho,\nu}(K-k,m;\pi'_k) \Big) \right)^2.$$

(b) If $\varepsilon_k = \varepsilon^{AE}$ for all $0 \leq k < K$, we have

$$C_{PI(AE),\rho,\nu}(K;r,s) = \left(\frac{1-\gamma}{2}\right)^2 \sup_{\pi'_0,\ldots,\pi'_K} \sum_{k=0}^{K-1} \alpha_k^{2(1-r)} \left(\sum_{m\geq 0} \gamma^m c_{PI_1,\rho,\nu}(K-k-1,m+1;\pi'_{k+1}) + \right.$$

$$\left. \sum_{m\geq 1} \gamma^m c_{PI_2,\rho,\nu}(K-k-1,m;\pi'_{k+1},\pi'_k) + c_{PI_3,\rho,\nu} \right)^2.$$

Proof. Part (a): Let $E_k = P^{\pi_{k+1}}(\mathbf{I} - \gamma P^{\pi_{k+1}})^{-1} - P^{\pi^*}(\mathbf{I} - \gamma P^{\pi_k})^{-1}$. It can be shown that (Munos [18, Lemma 4])

$$Q^* - Q^{\pi_{k+1}} \leq \gamma P^{\pi^*}(Q^* - Q^{\pi_k}) + \gamma E_k \varepsilon_k^{BR}.$$

By induction, we get

$$Q^* - Q^{\pi_K} \leq \gamma \sum_{k=0}^{K-1} (\gamma P^{\pi^*})^{K-k-1} E_k \varepsilon_k^{BR} + (\gamma P^{\pi^*})^K (Q^* - Q^{\pi_0}). \tag{4}$$

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Define $F_k = P^{\pi_{k+1}}(\mathbf{I} - \gamma P^{\pi_{k+1}})^{-1} + P^{\pi^*}(\mathbf{I} - \gamma P^{\pi_k})^{-1}$, and take point-wise absolute value of (4) to get

$$|Q^* - Q^{\pi_K}| \le \gamma \sum_{k=0}^{K-1} (\gamma P^{\pi^*})^{K-k-1} F_k |\varepsilon_k^{\mathrm{BR}}| + (\gamma P^{\pi^*})^K |Q^* - Q^{\pi_0}|.$$

Now we use α_k defined in (3) and introduce A_ks to simplify our further treatment:

$$A_k = \begin{cases} \frac{1-\gamma}{2}(P^{\pi^*})^{K-k-1} F_k & 0 \le k < K, \\ (P^{\pi^*})^K & k = K. \end{cases}$$

Note that $\sum_{k=0}^{K} \alpha_k = 1$, so Jensen inequality holds, i.e., $\phi(\sum_{k=0}^{K} a_k f_k) \le \sum_{k=0}^{K} a_k \phi(f_k)$ for convex $\phi(\cdot)$. Also it is shown in Lemma 12 of [7] that (1) $A_k : B(\mathcal{X} \times \mathcal{A}) \to B(\mathcal{X} \times \mathcal{A})$ are positive linear operators that satisfy $A_k \mathbf{1} = \mathbf{1}$, and (2) if $\phi(\cdot)$ is convex, $\phi(A_k Q) \le A_k(\phi(Q))$ where ϕ is applied point-wise.

Using these notations and noting that $Q^* - Q^{\pi_0} \le \frac{2}{1-\gamma} R_{\max} \mathbf{1}$ (where $\mathbf{1}$ is the constant function defined on domain $\mathcal{X}$ with the value of 1), we get

$$|Q^* - Q^{\pi_K}| \le \frac{2\gamma(1 - \gamma^{K+1})}{(1-\gamma)^2} \left[\sum_{k=0}^{K-1} \alpha_k A_k |\varepsilon_k^{\mathrm{BR}}| + \gamma^{-1} \alpha_K A_K R_{\max} \mathbf{1} \right]. \tag{5}$$

Denote $\lambda_K = [\frac{2\gamma(1-\gamma^{K+1})}{(1-\gamma)^2}]^p$. Take the pth power of both sides of (5) and apply Jensen inequality twice (once considering A_ks and once considering α_ks) to get

$$\|Q^* - Q^{\pi_K}\|_{p,\rho}^p = \frac{1}{|\mathcal{A}|} \sum_{a \in \mathcal{A}} \int_{\mathcal{X}} |Q^*(x,a) - Q^{\pi_K}(x,a)|^p \rho(dx) \le \lambda_K \rho \left[\sum_{k=0}^{K-1} \alpha_k A_k |\varepsilon_k^{\mathrm{BR}}|^p + \gamma^{-p} \alpha_K A_K R_{\max}^p \mathbf{1} \right].$$

Consider a term like $\rho A_k |\varepsilon_k^{\mathrm{BR}}|^p = \frac{1-\gamma}{2} \rho (P^{\pi^*})^{K-k-1} \left[P^{\pi_{k+1}}(\mathbf{I} - \gamma P^{\pi_{k+1}})^{-1} + P^{\pi^*}(\mathbf{I} - \gamma P^{\pi_k})^{-1} \right] |\varepsilon_k^{\mathrm{BR}}|^p$ for $0 \le k < K$. Expand $(\mathbf{I} - \gamma P^{\pi_{k+1}})^{-1}$ and $(\mathbf{I} - \gamma P^{\pi_k})^{-1}$ to have

$$\rho A_k |\varepsilon_k^{\mathrm{BR}}|^p = \frac{1-\gamma}{2} \rho \left[\sum_{m \ge 0} \gamma^m (P^{\pi^*})^{K-k-1} (P^{\pi_{k+1}})^{m+1} + \sum_{m \ge 0} \gamma^m (P^{\pi^*})^{K-k} (P^{\pi_k})^m \right] |\varepsilon_k^{\mathrm{BR}}|^p.$$

For any Borel measurable function $f : \mathcal{X} \to \mathbb{R}$, and probability measures μ_1 and μ_2 that satisfy $\mu_1 \ll \lambda$, $\mu_2 \ll \lambda$, and $\mu_1 \ll \mu_2$, where λ is the Lebesgue measure, we have the following Cauchy-Schwarz inequality:

$$\int_X f d\mu_1 \le \left(\int_{\mathcal{X}} \left| \frac{d\mu_1}{d\mu_2} \right|^2 d\mu_2 \right)^{\frac{1}{2}} \left(\int_{\mathcal{X}} f^2 d\mu_2 \right)^{\frac{1}{2}}.$$

Let us focus on a single term like $\rho(P^{\pi^*})^{K-k-1}(P^{\pi_{k+1}})^{m+1}|\varepsilon_k^{\mathrm{BR}}|^p$, and apply the Cauchy-Schwarz inequality to it. We have

$$\rho(P^{\pi^*})^{K-k-1}(P^{\pi_{k+1}})^{m+1}|\varepsilon_k^{\mathrm{BR}}|^p \le \left(\int_{\mathcal{X}} \left| \frac{d \left(\rho(P^{\pi^*})^{K-k-1}(P^{\pi_{k+1}})^{m+1} \right)}{d\nu} \right|^2 d\nu \right)^{\frac{1}{2}} \left(\int_{\mathcal{X}} |\varepsilon_k^{\mathrm{BR}}|^{2p} d\nu \right)^{\frac{1}{2}}$$

$$= c_{\mathrm{PI}_1,\rho,\nu}(K-k-1, m+1; \pi_{k+1}) \left\| \varepsilon_k^{\mathrm{BR}} \right\|_{2p,\nu}^p.$$

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Doing the same for the other terms $(P^{\pi^*})^{K-k}(P^{\pi_k})^m$, and noting that $\rho A_K \mathbf{1} = \rho \mathbf{1} = 1$ implies that

$$\|Q^* - Q^{\pi_K}\|_{p,\rho}^p \leq$$

$$\lambda_K \left[\frac{1-\gamma}{2} \sum_{k=0}^{K-1} \alpha_k \sum_{m \geq 0} \gamma^m \left(c_{\mathrm{PI}_1,\rho,\nu}(K-k-1, m+1; \pi_{k+1}) + c_{\mathrm{PI}_1,\rho,\nu}(K-k, m; \pi_k) \right) \left\| \varepsilon_k^{\mathrm{BR}} \right\|_{2p,\nu}^p \right.$$

$$\left. + \gamma^{-p} \alpha_K R_{\max}^p \right].$$

In order to separate concentrability coefficients and $\{\varepsilon_k^{\mathrm{BR}}\}_{k=0}^{K-1}$, we use Hölder inequality $\sum_{k=0}^{K-1} a_k b_k \leq (\sum_{k=0}^{K-1} |a_k|^s)^{\frac{1}{s}} (\sum_{k=0}^{K-1} |b_k|^{s'})^{\frac{1}{s'}}$ with $s \in (1, \infty)$ and $\frac{1}{s} + \frac{1}{s'} = 1$. Let $a_k = \alpha_k^r \left\| \varepsilon_k^{\mathrm{BR}} \right\|_{2p,\nu}^p$ and $b_k = \alpha_k^{1-r} \sum_{m \geq 0} \gamma^m \left(c_{\mathrm{PI}_1,\rho,\nu}(K-k-1, m+1; \pi_{k+1}) + c_{\mathrm{PI}_1,\rho,\nu}(K-k, m; \pi_k) \right)$ for some $r \in [0,1]$. Therefore for all $(s,r) \in (1, \infty) \times [0,1]$, we have

$$\|Q^* - Q^{\pi_K}\|_{p,\rho}^p \leq \lambda_K \frac{1-\gamma}{2} \left[\sum_{k=0}^{K-1} \alpha_k^{s(1-r)} \left(\sum_{m \geq 0} \gamma^m \left(c_{\mathrm{PI}_1,\rho,\nu}(K-k-1, m+1; \pi_{k+1}) + c_{\mathrm{PI}_1,\rho,\nu}(K-k, m; \pi_k) \right) \right)^s \right]^{\frac{1}{s}}$$

$$\times \left[\sum_{k=0}^{K-1} \alpha_k^{s'r} \left\| \varepsilon_k^{\mathrm{BR}} \right\|_{2p,\nu}^{ps'} \right]^{\frac{1}{s'}} + \lambda_K \gamma^{-p} \alpha_K R_{\max}^p. \tag{6}$$

Because $\{\pi_k\}_{k=0}^K$ are not known, we take a supremum over all policies. Moreover as (6) holds for all $(s,r) \in (1,\infty) \times [0,1]$, we may take the infimum of the right hand side. Also note that $\frac{1-\gamma}{1-\gamma^{K+1}} < 1$ and $\lambda_K \leq [\frac{2\gamma}{(1-\gamma)^2}]^p$. After taking the pth root, we have

$$\|Q^* - Q^{\pi_K}\|_{p,\rho} \leq \frac{2\gamma}{(1-\gamma)^2} \left[\inf_{(s,r) \in (1,\infty) \times [0,1]} C_{\mathrm{PI},\rho,\nu}^{\frac{1}{ps}}(K; r, s) \mathcal{E}^{\frac{1}{ps'}} (\varepsilon_0^{\mathrm{BR}}, \dots, \varepsilon_{K-1}^{\mathrm{BR}}; r, s) + \gamma^{\frac{K}{p}-1} R_{\max} \right],$$

where

$$C_{\mathrm{PI(BR)},\rho,\nu}(K; r) = (\frac{1-\gamma}{2})^2 \sup_{\pi_0', \dots, \pi_K'} \sum_{k=0}^{K-1} \alpha_k^{2(1-r)} \left(\sum_{m \geq 0} \gamma^m \left(c_{\mathrm{PI}_1,\rho,\nu}(K-k-1, m+1; \pi_{k+1}') + \right. \right.$$

$$\left. \left. c_{\mathrm{PI}_1,\rho,\nu}(K-k, m; \pi_k') \right) \right)^2,$$

and $\mathcal{E}(\varepsilon_0^{\mathrm{BR}}, \dots, \varepsilon_{K-1}^{\mathrm{BR}}; r, s) = \sum_{k=0}^{K-1} \alpha_k^{s'r} \left\| \varepsilon_k^{\mathrm{BR}} \right\|_{2p,\nu}^{ps'}$.

This result is general and holds for all $s \in (0,1)$. In order to make it more accessible, but at the cost of loosening of the upper bound, we set $s = s' = 2$. This finishes the proof of Part (a).

Part (b): The proof of this part is similar to the proof of Part (a). We briefly sketch the key steps: Define $E_k = P^{\pi_{k+1}}(\mathbf{I} - \gamma P^{\pi_{k+1}})^{-1}(\mathbf{I} - \gamma P^{\pi_k}) - P^{\pi^*}$. From Munos [18, Lemma 4] one can show that

$$Q^* - Q^{\pi_K} \leq \gamma \sum_{k=0}^{K-1} (\gamma P^{\pi^*})^{K-k-1} E_k \varepsilon_k^{\mathrm{AE}} + (\gamma P^{\pi^*})^K (Q^* - Q^{\pi_0}). \tag{7}$$

Define $F_k = P^{\pi_{k+1}}(\mathbf{I} - \gamma P^{\pi_{k+1}})^{-1}(\mathbf{I} - \gamma P^{\pi_k}) + P^{\pi^*}$ and take point-wise absolute value of (7) and use the same definition of A_ks as Part (a) (with the new F_ks) to get

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$$|Q^* - Q^{\pi_K}| \le \frac{2\gamma(1 - \gamma^{K+1})}{(1-\gamma)^2} \left[\sum_{k=0}^{K-1} \alpha_k A_k |\varepsilon_k^{\text{AE}}| + \gamma^{-1}\alpha_K A_K R_{\max}\mathbf{1} \right].$$

Consider a term like $\rho A_k |\varepsilon_k^{\text{BR}}|^p$ for $0 \le k < K$ and expand $(\mathbf{I} - \gamma P^{\pi_{k+1}})^{-1}$.

$$\rho A_k |\varepsilon_k^{\text{AE}}|^p = \frac{1-\gamma}{2}\rho \left[\sum_{m\ge 0} \gamma^m (P^{\pi^*})^{K-k-1}(P^{\pi_{k+1}})^{m+1}(\mathbf{I} - \gamma P^{\pi_k}) + P^{\pi^*} \right] |\varepsilon_k^{\text{AE}}|^p.$$

After performing the same change of measure and applying the Cauchy-Schwarz inequality, we have
$\|Q^* - Q^{\pi_K}\|_{p,\rho}^p \le$

$$\lambda_K \left[\frac{1-\gamma}{2} \sum_{k=0}^{K-1} \alpha_k \left(\sum_{m \ge 0} \gamma^m c_{\text{PI}_1,\rho,\nu}(K-k-1, m+1; \pi_{k+1}) + \right.\right.$$

$$\left.\left. \sum_{m \ge 1} \gamma^m c_{\text{PI}_2,\rho,\nu}(K-k-1, m; \pi_{k+1}, \pi_k) + c_{\text{PI}_3,\rho,\nu} \right) \|\varepsilon_k^{\text{AE}}\|_{2p,\nu}^p \right.$$

$$\left. + \gamma^{-p}\alpha_K R_{\max}^p \right].$$

Application of Hölder inequality with a similarly defined decomposition and then taking supremum over policies leads to

$$\|Q^* - Q^{\pi_K}\|_{p,\rho} \le \frac{2\gamma}{(1-\gamma)^2} \left[\inf_{(s,r)\in(1,\infty)\times[0,1]} C_{\text{PI},\rho,\nu}^{\frac{1}{ps}}(K; r, s) \mathcal{E}^{\frac{1}{ps'}}(\varepsilon_0^{\text{AE}}, \ldots, \varepsilon_{K-1}^{\text{AE}}; r, s) + \gamma^{\frac{K}{p}-1} R_{\max} \right],$$

where

$$C_{\text{PI(AE)},\rho,\nu}(K; r, s) = \left(\frac{1-\gamma}{2}\right)^s \sup_{\pi_0',\ldots,\pi_K'} \sum_{k=0}^{K-1} \alpha_k^{s(1-r)} \left(\sum_{m\ge 0} \gamma^m c_{\text{PI}_1,\rho,\nu}(K-k-1, m+1; \pi_{k+1}') + \right.$$

$$\left. \sum_{m\ge 1} \gamma^m c_{\text{PI}_2,\rho,\nu}(K-k-1, m; \pi_{k+1}', \pi_k') + c_{\text{PI}_3,\rho,\nu} \right)^s$$

and $\mathcal{E}(\varepsilon_0^{\text{AE}}, \ldots, \varepsilon_{K-1}^{\text{AE}}; r, s) = \sum_{k=0}^{K-1} \alpha_k^{s'r} \|\varepsilon_k^{\text{AE}}\|_{2p,\nu}^{ps'}$. $\qquad\square$

4 Approximate Value Iteration

Consider the AVI procedure and the sequence $V_0 \to V_1 \to \cdots \to V_{K-1}$, in which V_{k+1} is the result of approximately applying the Bellman optimality operator on the previous estimate V_k, i.e., $V_{k+1} \approx T^* V_k$. Denote the approximation error caused at each iteration by

$$\varepsilon_k = T^* V_k - V_{k+1}. \tag{8}$$

The goal of this section is to analyze AVI procedure and to relate the approximation error sequence $\{\varepsilon_k\}_{k=0}^{K-1}$ to the performance loss $\|V^* - V^{\pi_K}\|_{p,\rho}$ of the obtained policy π_K, which is the greedy policy w.r.t. V_{K-1}.

Theorem 4 (Error Propagation for AVI). *Let $p \ge 1$ be a real number, K be a positive integer, and $V_{max} \le \frac{R_{max}}{1-\gamma}$. Then for any sequence $\{V_k\}_{k=0}^{K-1} \subset B(\mathcal{X}, V_{max})$, and the corresponding sequence $\{\varepsilon_k\}_{k=0}^{K-1}$ defined in (8), we have*

$$\|V^* - V^{\pi_K}\|_{p,\rho} \le \frac{2\gamma}{(1-\gamma)^2} \left[\inf_{r\in[0,1]} C_{VI,\rho,\nu}^{\frac{1}{2p}}(K; r) \mathcal{E}^{\frac{1}{2p}}(\varepsilon_0, \ldots, \varepsilon_{K-1}; r) + \frac{2}{1-\gamma}\gamma^{\frac{K}{p}} R_{max} \right],$$

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where

$$C_{VI,\rho,\nu}(K;r) = (\frac{1-\gamma}{2})^2 \sup_{\pi'} \sum_{k=0}^{K-1} \alpha_k^{2(1-r)} \left(\sum_{m\geq 0} \gamma^m \left(c_{VI,\rho,\nu}(m, K-k; \pi') + c_{VI,\rho,\nu}(m+1, K-k-1; \pi') \right) \right)^2 ,$$

and $\mathcal{E}(\varepsilon_0, \ldots, \varepsilon_{K-1}; r) = \sum_{k=0}^{K-1} \alpha_k^{2r} \|\varepsilon_k\|_{2p,\nu}^{2p}$.

Proof. First we derive a point-wise bound relating $V^* - V^{\pi_K}$ to $\{\varepsilon_k\}_{k=0}^{K-1}$ similar to Lemma 4.1 of Munos [17].

$$V^* - V_{k+1} = T^{\pi^*} V^* - T^{\pi^*} V_k + T^{\pi^*} V_k - T^* V_k + \varepsilon_k \leq \gamma P^{\pi^*}(V^* - V_k) + \varepsilon_k$$

where we used the property of the Bellman optimality operator $T^* V_k \geq T^{\pi^*} V_k$ and the definition of ε_k (8). Induction results in

$$V^* - V_K \leq \sum_{k=0}^{K-1} \gamma^{K-k-1} (P^{\pi^*})^{K-k-1} \varepsilon_k + \gamma^K (P^{\pi^*})^K (V^* - V_0). \tag{9}$$

Benefiting from $T^* V_K \geq T^{\pi^*} V_K$ and noting that $T^* V_K = T^{\pi_K} V^{\pi_K}$ by definition of the greedy policy,

$$
\begin{aligned}
V^* - V^{\pi_K} &= T^{\pi^*} V^* - T^{\pi^*} V_K + T^{\pi^*} V_K - T^* V_K + T^* V_K - T^{\pi_K} V^{\pi_K} \\
&\leq T^{\pi^*} V^* - T^{\pi^*} V_K + T^* V_K - T^{\pi_K} V^{\pi_K} \\
&= \gamma P^{\pi^*}(V^* - V_K) + \gamma P^{\pi_K}(V_K - V^{\pi_K}) \\
&= \gamma P^{\pi^*}(V^* - V_K) + \gamma P^{\pi_K}(V_K - V^* + V^* + V^{\pi_K}).
\end{aligned}
$$

Re-arranging and using Lemma 4.2 of [17], we deduce that

$$V^* - V^{\pi_K} \leq \gamma (\mathbf{I} - \gamma P^{\pi_K})^{-1} (P^{\pi^*} - P^{\pi_K})(V^* - V_K) \tag{10}$$

Plugging (9) in (10) and taking the absolute value of both sides, we get the following point-wise inequality

$$
V^* - V^{\pi_K} \leq \gamma (\mathbf{I} - \gamma P^{\pi_K})^{-1} \Bigg[\sum_{k=0}^{K-1} \gamma^{K-k-1} \left((P^{\pi^*})^{K-k} + P^{\pi_K}(P^{\pi^*})^{K-k-1} \right) |\varepsilon_k|
$$

$$
+ \gamma^K \left((P^{\pi^*})^{K+1} + P^{\pi_K}(P^{\pi^*})^K \right) |V^* - V_0| \Bigg] \tag{11}
$$

As in the proof of Theorem 3, we use α_ks as defined in (3) and introduce

$$
A_k = \begin{cases} \frac{1-\gamma}{2}(\mathbf{I} - \gamma P^{\pi_K})^{-1} \left[(P^{\pi^*})^{K-k} + P^{\pi_K}(P^{\pi^*})^{K-k-1} \right] & 0 \leq k < K, \\ \frac{1-\gamma}{2}(\mathbf{I} - \gamma P^{\pi_K})^{-1} \left((P^{\pi^*})^{K+1} + P^{\pi_K}(P^{\pi^*})^K \right) & k = K. \end{cases}
$$

Note that we use the same α_ks as in the proof of Theorem 3, but A_ks are different. Nevertheless, they satisfy the same properties that allow us to apply Jensen inequality. We have $|V^* - V_0| \leq \frac{2}{1-\gamma} R_{\max} \mathbf{1}$ to get

$$
V^* - V^{\pi_K} \leq \frac{2\gamma(1-\gamma^{K+1})}{(1-\gamma)^2} \left[\sum_{k=0}^{K-1} \alpha_k A_k |\varepsilon_k| + \alpha_K A_K \frac{2}{1-\gamma} R_{\max} \mathbf{1} \right].
$$

Now take the pth power of both sides of (11), and apply Jensen inequality twice (once considering A_ks and once considering α_ks),

$$
\|V^* - V^{\pi_K}\|_{p,\rho}^p = \lambda_K \rho \left[\sum_{k=0}^{K-1} \alpha_k A_k |\varepsilon_k|^p + \alpha_K \left(\frac{2}{1-\gamma} \right)^p A_k R_{\max}^p \mathbf{1} \right].
$$

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Consider a term like $\rho A_k |\varepsilon_k|^p$ for $0 \le k < K$:

$$\rho A_k |\varepsilon_k|^p = \frac{1-\gamma}{2}(\mathbf{I} - \gamma P^{\pi_K})^{-1} \left[(P^{\pi^*})^{K-k} + P^{\pi_K}(P^{\pi^*})^{K-k-1} \right] |\varepsilon_k|^p$$

$$= \frac{1-\gamma}{2}\rho \left[\sum_{m \ge 0} \gamma^m \left((P^{\pi_K})^m (P^{\pi^*})^{K-k} + (P^{\pi_K})^{m+1}(P^{\pi^*})^{K-k-1} \right) \right] |\varepsilon_k|^p.$$

Applying Cauchy-Schwarz inequality, as we did in Theorem 3, we deduce

$$\|V^* - V^{\pi_K}\|_{p,\rho}^p \le \lambda_K \left[\frac{1-\gamma}{2} \sum_{k=0}^{K-1} \alpha_k \sum_{m \ge 0} \gamma^m \left(c_{\mathrm{VI},\rho,\nu}(m, K-k; \pi_K) + c_{\mathrm{VI},\rho,\nu}(m+1, K-k-1; \pi_K) \right) \|\varepsilon_k\|_{2p,\nu}^p \right.$$

$$\left. + \alpha_K \left(\frac{2}{1-\gamma} \right)^p R_{\max}^p \right].$$

Use Hölder inequality with $a_k = \alpha_k^r \|\varepsilon_k\|_{2p,\nu}^p$ and $b_k = \alpha_k^{1-r} \sum_{m \ge 0} \gamma^m \left(c_{\mathrm{VI},\rho,\nu}(m, K-k; \pi_K) + c_{\mathrm{VI},\rho,\nu}(m+1, K-k-1; \pi_K) \right)$ (all variables are defined the same as in the proof of Theorem 3). Therefore for all $(s,r) \in (1, \infty) \times [0,1]$, we have

$$\|V^* - V^{\pi_K}\|_{p,\rho} \le \frac{2\gamma}{(1-\gamma)^2} \left[\inf_{(s,r) \in (1,\infty) \times [0,1]} C_{\mathrm{VI},\rho,\nu}^{\frac{1}{ps}}(K; r, s) \mathcal{E}^{\frac{1}{ps'}}(\varepsilon_0, \ldots, \varepsilon_{K-1}; r, s) + \frac{2}{1-\gamma} \gamma^{\frac{K}{p}} R_{\max} \right],$$

where

$$C_{\mathrm{VI},\rho,\nu}(K; r, s) = (\frac{1-\gamma}{2})^s \sup_{\pi'} \sum_{k=0}^{K-1} \alpha_k^{s(1-r)} \left(\sum_{m \ge 0} \gamma^m \left(c_{\mathrm{VI},\rho,\nu}(m, K-k; \pi') + c_{\mathrm{VI},\rho,\nu}(m+1, K-k-1; \pi') \right) \right)^s,$$

and $\mathcal{E}(\varepsilon_0, \ldots, \varepsilon_{K-1}; r, s) = \sum_{k=0}^{K-1} \alpha_k^{s'r} \|\varepsilon_k\|_{2p,\nu}^{ps'}$. To simplify the bound, at the cost of loosening the upper bound, we set $s = s' = 2$.

$\square$

5 Discussion

In this section, we discuss significant improvements of Theorems 3 and 4 over previous results such as [16, 18, 17, 7].

5.1 L_p norm instead of L_∞ norm

As opposed to most error upper bounds, Theorems 3 and 4 relate $\|V^* - V^{\pi_K}\|_{p,\rho}$ to the L_p norm of the approximation or Bellman errors $\|\varepsilon_k\|_{2p,\nu}$ of iterations in API/AVI. This should be contrasted with the traditional, and more conservative, results such as $\limsup_{k \to \infty} \|V^* - V^{\pi_k}\|_\infty \le \frac{2\gamma}{(1-\gamma)^2} \limsup_{k \to \infty} \|V^{\pi_k} - V_k\|_\infty$ for API (Proposition 6.2 of Bertsekas and Tsitsiklis [16]). The use of L_p norm not only is a huge improvement over conservative supremum norm, but also allows us to benefit from the vast literature on supervised learning techniques, which usually provides error upper bounds in the form of L_p norms, in the context of RL/Planning problems. This is especially interesting for the case of $p = 1$ as the performance loss $\|V^* - V^{\pi_K}\|_{1,\rho}$ is the difference between the expected return of the optimal policy and the resulted policy π_K when the initial state distribution is ρ. Convenient enough, the errors appearing in the upper bound are in the form of $\|\varepsilon_k\|_{2,\nu}$ which is very common in the supervised learning literature. This type of improvement, however, has been done in the past couple of years [18, 17, 7] - see Proposition 1 in Section 1.

5.2 Expected versus supremum concentrability of the future-state distribution

The concentrability coefficients (Definition 2) reflect the effect of future-state distribution on the performance loss $\|V^* - V^{\pi_K}\|_{p,\rho}$. Previously it was thought that the key contributing factor to the performance loss is the supremum of the Radon-Nikodym derivative of these two distributions. This is

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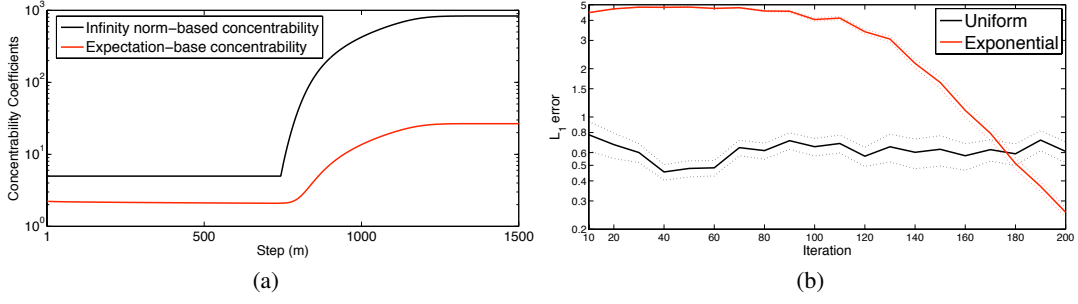
Figure 1: (a) Comparison of $\mathbb{E}_{X\sim\nu}\left[\left|\frac{d(\rho(P^\pi)^m)}{d\nu}\right|^2\right]^{\frac{1}{2}}$ and $\left\|\frac{d(\rho(P^\pi)^m)}{d\nu}\right\|_\infty$ (b) Comparison of $\|Q^* - Q_k\|_1$ for uniform and exponential data sampling schedule. The total number of samples is the same. [The Y-scale of both plots is logarithmic.]

evident in the definition of $C_{\rho,\nu}$ in Proposition 1 where we have terms in the form of $||\frac{d(\rho(P^\pi)^m)}{d\nu}||_\infty$ instead of $\left(\mathbb{E}_{X\sim\nu}\left[\left|\frac{d(\rho(P^\pi)^m)}{d\nu}(X)\right|^2\right]\right)^{\frac{1}{2}}$ that we have in Definition 2.

Nevertheless, it turns out that the key contributing factor that determines the performance loss is the *expectation* of the squared Radon-Nikodym derivative instead of its supremum. Intuitively this implies that even if for some subset of $\mathcal{X}' \subset \mathcal{X}$ the ratio $\frac{d(\rho(P^\pi)^m)}{d\nu}$ is large but the probability $\nu(\mathcal{X}')$ is very small, performance loss due to it is still small. This phenomenon has not been suggested by previous results.

As an illustration of this difference, consider a Chain Walk with 1000 states with a single policy that drifts toward state 1 of the chain. We start with $\rho(x) = \frac{1}{201}$ for $x \in [400, 600]$ and zero everywhere else. Then we evaluate both $||\frac{d(\rho(P^\pi)^m)}{d\nu}||_\infty$ and $(\mathbb{E}_{X\sim\nu}\left[\left|\frac{d(\rho(P^\pi)^m)}{d\nu}\right|^2\right])^{\frac{1}{2}}$ for $m = 1, 2, \ldots$ when ν is the uniform distribution. The result is shown in Figure 1a. One sees that the ratio is constant in the beginning, but increases when the distribution $\rho(P^\pi)^m$ concentrates around state 1, until it reaches steady-state. The growth and the final value of the expectation-based concentrability coefficient is much smaller than that of supremum-based.

It is easy to show that if the Chain Walk has N states and the policy has the same concentrating behavior and ν is uniform, then $||\frac{d(\rho(P^\pi)^m)}{d\nu}||_\infty \to N$, while $(\mathbb{E}_{X\sim\nu}\left[\left|\frac{d(\rho(P^\pi)^m)}{d\nu}\right|^2\right])^{\frac{1}{2}} \to \sqrt{N}$ when $m \to \infty$. The ratio, therefore, would be of order $\Theta(\sqrt{N})$. This clearly shows the improvement of this new analysis in a simple problem. One may anticipate that this sharper behavior happens in many other problems too.

More generally, consider $C_\infty = ||\frac{d\mu}{d\nu}||_\infty$ and $C_{L_2} = (\mathbb{E}_{X\sim\nu}\left[\left|\frac{d\mu}{d\nu}\right|^2\right])^{\frac{1}{2}}$. For a finite state space with N states and ν is the uniform distribution, $C_\infty \leq N$ but $C_{L_2} \leq \sqrt{N}$. Neglecting all other differences between our results and the previous ones, we get a performance upper bound in the form of $\|Q^* - Q^{\pi_K}\|_{1,\rho} \leq c_1(\gamma)O(N^{1/4})\sup_k \|\varepsilon_k\|_{2,\nu}$, while Proposition 1 implies that $\|Q^* - Q^{\pi_K}\|_{1,\rho} \leq c_2(\gamma)O(N^{1/2})\sup_k \|\epsilon_k\|_{2,\nu}$. This difference between $O(N^{1/4})$ and $O(N^{1/2})$ shows a significant improvement.

5.3 Error decaying property

Theorems 3 and 4 show that the dependence of performance loss $\|V^* - V^{\pi_K}\|_{p,\rho}$ (or $\|Q^* - Q^{\pi_K}\|_{p,\rho}$) on $\{\varepsilon_k\}_{k=0}^{K-1}$ is in the form of $\mathcal{E}(\varepsilon_0, \ldots, \varepsilon_{K-1}; r) = \sum_{k=0}^{K-1} \alpha_k^{2r} \|\varepsilon_k\|_{2p,\nu}^{2p}$. This has a very special structure in that the approximation errors at later iterations have more contribution to the final performance loss. This behavior is obscure in previous results such as [17, 7] that the dependence of the final performance loss is expressed as $\mathcal{E}(\varepsilon_0, \ldots, \varepsilon_{K-1}; r) = \max_{k=0,\ldots,K-1} \|\varepsilon_k\|_{p,\nu}$ (see Proposition 1).

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This property has practical and algorithmic implications too. It says that it is better to put more effort on having a lower Bellman or approximation error at later iterations of API/AVI. This, for instance, can be done by gradually increasing the number of samples throughout iterations, or to use more powerful, and possibly computationally more expensive, function approximators for the later iterations of API/AVI.

To illustrate this property, we compare two different sampling schedules on a simple MDP. The MDP is a 100-state, 2-action chain similar to Chain Walk problem in the work of Lagoudakis and Parr [5]. We use AVI with a lookup-table function representation. In the first sampling schedule, every 20 iterations we generate a fixed number of *fresh* samples by following a uniformly random walk on the chain (this means that we throw away old samples). This is the *fixed* strategy. In the *exponential* strategy, we again generate new samples every 20 iterations but the number of samples at the kth iteration is ck^γ. The constant c is tuned such that the total number of both sampling strategy is almost the same (we give a slight margin of about 0.1% of samples in favor of the fixed strategy). What we compare is $\|Q^* - Q_k\|_{1,\nu}$ when ν is the uniform distribution. The result can be seen in Figure 1b. The improvement of the exponential sampling schedule is evident. Of course, one may think of more sophisticated sampling schedules but this simple illustration should be sufficient to attract the attention of practitioners to this phenomenon.

5.4 Restricted search over policy space

One interesting feature of our results is that it puts more structure and restriction on the way policies may be selected. Comparing $C_{\text{PI},\rho,\nu}(K;r)$ (Theorem 3) and $C_{\text{VI},\rho,\nu}(K;r)$ (Theorem 4) with $C_{\rho,\nu}$ (Proposition 1) we see that:

(1) Each concentrability coefficient in the definition of $C_{\text{PI},\rho,\nu}(K;r)$ depends only on a single or two policies (e.g., π'_k in $c_{\text{PI}_1,\rho,\nu}(K-k,m;\pi'_k)$). The same is true for $C_{\text{VI},\rho,\nu}(K;r)$. In contrast, the mth term in $C_{\rho,\nu}$ has $\pi_1,\ldots,\pi_m$ as degrees of freedom, and this number is growing as $m \to \infty$.

(2) The operator sup in $C_{\text{PI},\rho,\nu}$ and $C_{\text{VI},\rho,\nu}$ appears *outside* the summation. Because of that, we only have $K+1$ degrees of freedom $\pi'_0,\ldots,\pi'_K$ to choose from in API and remarkably only a single degree of freedom in AVI. On the other other hand, sup appears *inside* the summation in the definition of $C_{\rho,\nu}$. One may construct an MDP that this difference in the ordering of sup leads to an arbitrarily large ratio of two different ways of defining the concentrability coefficients.

(3) In API, the definitions of concentrability coefficients $c_{\text{PI}_1,\rho,\nu}$, $c_{\text{PI}_2,\rho,\nu}$, and $c_{\text{PI}_3,\rho,\nu}$ (Definition 2) imply that if $\rho = \rho^*$, the stationary distribution induced by an optimal policy π^*, then $c_{\text{PI}_1,\rho,\nu}(m_1,m_2;\pi) = c_{\text{PI}_1,\rho,\nu}(\cdot,m_2;\pi) = (\mathbb{E}_{X\sim\nu}\left[\left|\frac{d(\rho^*(P^\pi)^{m_2})}{d\nu}\right|^2\right])^{\frac{1}{2}}$ (similar for the other two coefficients). This special structure is hidden in the definition of $C_{\rho,\nu}$ in Proposition 1, and instead we have an extra m_1 degrees of flexibility.

Remark 1. *For general MDPs, the computation of concentrability coefficients in Definition 2 is difficult, as it is for similar coefficients defined in [18, 17, 7].*

6 Conclusion

To analyze an API/AVI algorithm and to study its statistical properties such as consistency or convergence rate, we require to (1) analyze the statistical properties of the algorithm running at each iteration, and (2) study the way the policy approximation/Bellman errors propagate and influence the quality of the resulted policy.

The analysis in the first step heavily uses tools from the Statistical Learning Theory (SLT) literature, e.g., Györfi et al. [22]. In some cases, such as AVI, the problem can be cast as a standard regression with the twist that extra care should be taken to the temporal dependency of data in RL scenario. The situation is a bit more complicated for API methods that directly aim for the fixed-point solution (such as LSTD and its variants), but still the same kind of tools from SLT can be used too – see Antos et al. [7], Maillard et al. [8].

The analysis for the second step is what this work has been about. In our Theorems 3 and 4, we have provided upper bounds that relate the errors at each iteration of API/AVI to the performance

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loss of the whole procedure. These bounds are qualitatively tighter than the previous results such as those reported by [18, 17, 7], and provide a better understanding of what factors contribute to the difficulty of the problem. In Section 5, we discussed the significance of these new results and the way they improve previous ones.

Finally, we should note that there are still some unaddressed issues. Perhaps the most important one is to study the behavior of concentrability coefficients $c_{\mathrm{PI}_1,\rho,\nu}(m_1, m_2; \pi)$, $c_{\mathrm{PI}_2,\rho,\nu}(m_1, m_2; \pi_1, \pi_2)$, and $c_{\mathrm{VI},\rho,\nu}(m_1, m_2; \pi)$ as a function of m_1, m_2, and of course the transition kernel P of MDP. A better understanding of this question alongside a good understanding of the way each term ε_k in $\mathcal{E}(\varepsilon_0, \ldots, \varepsilon_{K-1}; r)$ behaves, help us gain more insight about the error convergence behavior of the RL/Planning algorithms.

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