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SENSITIVITY ANALYSIS USING ITÔ–MALLIAVIN CALCULUS AND MARTINGALES, AND APPLICATION TO STOCHASTIC OPTIMAL CONTROL*

EMMANUEL GOBET† AND RÉMI MUNOS†

Abstract. We consider a multidimensional diffusion process $(X_t^\alpha)_{0\leq t\leq T}$ whose dynamics depends on a parameter α. Our first purpose is to write as an expectation the sensitivity $\nabla_\alpha J(\alpha)$ for the expected cost $J(\alpha) = \mathbb{E}(f(X_T^\alpha))$, in order to evaluate it using Monte Carlo simulations. This issue arises, for example, from stochastic control problems (where the controller is parameterized, which reduces the control problem to a parametric optimization one) or from model misspecifications in finance. Previous evaluations of $\nabla_\alpha J(\alpha)$ using simulations were limited to smooth cost functions f or to diffusion coefficients not depending on α (see Yang and Kushner, *SIAM J. Control Optim.*, 29 (1991), pp. 1216–1249). In this paper, we cover the general case, deriving three new approaches to evaluate $\nabla_\alpha J(\alpha)$, which we call the *Malliavin calculus approach*, the *adjoint approach*, and the *martingale approach*. To accomplish this, we leverage Itô calculus, Malliavin calculus, and martingale arguments. In the second part of this work, we provide discretization procedures to simulate the relevant random variables; then we analyze their respective errors. This analysis proves that the discretization error is essentially linear with respect to the time step. This result, which was already known in some specific situations, appears to be true in this much wider context. Finally, we provide numerical experiments in random mechanics and finance and compare the different methods in terms of variance, complexity, computational time, and time discretization error.

Key words. sensitivity analysis, parameterized control, Malliavin calculus, weak approximation

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1. Introduction.

We consider a d-dimensional stochastic differential equation (SDE) defined by

$$X_t = x + \int_0^t b(s, X_s, \alpha)\, ds + \sum_{j=1}^{q} \int_0^t \sigma_j(s, X_s, \alpha)\, dW_s^j, \tag{1.1}$$

where α is a parameter (taking values in $\mathcal{A} \subset \mathbb{R}^m$) and $(W_t)_{0\leq t\leq T}$ is a standard Brownian motion in $\mathbb{R}^q$ on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, \mathbb{P})$, with the usual assumptions on the filtration $(\mathcal{F}_t)_{0\leq t\leq T}$.

We first aim at evaluating the sensitivity w.r.t. α of the expected cost

$$J(\alpha) = \mathbb{E}\left(f(X_T)\right), \tag{1.2}$$

for a given terminal cost f and for a fixed time T. The sensitivity of more general functionals including instantaneous costs like $\mathbb{E}\big(\int_0^T g(t, X_t)dt + f(X_T)\big) = \int_0^T \mathbb{E}(g(t, X_t))dt + \mathbb{E}(f(X_T))$ will follow by discretizing the integral and applying the sensitivity estimator for each time.

This evaluation is a typical issue raised in various applications. A first example is the analysis of the impact on the expected cost $J(\alpha)$ of a misspecification of a stochastic model (defined by a SDE with coefficients $\bar{b}(t, x)$ and $(\bar{\sigma}_j(t, x))_{1\leq j\leq q}$). The issue

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†Centre de Mathématiques Appliquées, Ecole Polytechnique, 91128 Palaiseau Cedex, France (emmanuel.gobet@polytechnique.fr, remi.munos@polytechnique.fr).

may be formulated by setting $b(t,x,\alpha) = \bar{b}(t,x) + \sum_{i=1}^{m} \alpha_i \phi_i(t,x)$ (and analogously for $(\sigma_j(t,x,\alpha))_{1\leq j\leq q}$), then computing the sensitivities at the point $\alpha = 0$. In finance, misspecifications in option pricing procedures usually concern the diffusion coefficients $(\bar{\sigma}_j(t,x))_{1\leq j\leq q}$ (the *volatility* of the assets). There are also some connections with the so-called model risk problem (see Cvitanić and Karatzas [CK99]).

Stochastic control is another field requiring sensitivity analysis. For instance, if the controlled SDE is defined by $dX_t = \bar{b}(t,X_t,u_t)\ dt + \sum_{j=1}^{q} \bar{\sigma}_j(t,X_t,u_t) dW_t^j$, the problem is to find the maximal value of $\mathbb{E}(f(X_T))$ among the admissible policies $(u_t)_{0\leq t\leq T}$. In low dimensions (say 1 or 2), numerical methods based on the dynamic programming principle can be successfully implemented (see Kushner and Dupuis [KD01] for some references), but they become inefficient in higher dimensions. Alternatively, one can use *policy search* algorithms (see [BB01] and references therein). It consists in seeking a *good* policy in a feedback form using a parametric representation, that is, $u_t = u(t,X_t,\alpha)$: in that case, one puts $b(t,x,\alpha) = \bar{b}(t,x,u(t,x,\alpha))$ and $\sigma_j(t,x,\alpha) = \bar{\sigma}_j(t,x,u(t,x,\alpha))$. The policy function $u(t,x,\alpha)$ can be parameterized through a linear approximation (a linear combination of basis functions) or through a nonlinear one (e.g., with neural networks, see Rumelhart and McClelland [RM86] or Haykin [Hay94] for general references). Then, one might use a standard parametric optimization procedure such as the stochastic gradient method or other stochastic approximation algorithms (see Polyak [Pol87]; Benveniste, Metivier, and Priouret [BMP90]; Kushner and Yin [KY97]), which require sensitivity estimations of $J(\alpha)$ w.r.t. α, such as $\nabla_\alpha J(\alpha)$. This gradient is the quantity we will focus on in this paper.

Since the setting is a priori multidimensional, we propose a Monte Carlo approach for the numerical computations. The evaluation of $J(\alpha)$ is standard and has been widely studied. For an introduction to numerical approximations of SDEs, we refer the reader to Kloeden and Platen [KP95], for instance. To our knowledge, there are three different approaches to compute $\nabla_\alpha J(\alpha)$ in our context:

1. The *resampling method* (see Glasserman and Yao [GY92], L'Ecuyer and Perron [LP94] for instance), which consists in computing different values of $J(\alpha)$ for some close values of the parameter α and then forming some appropriate differences to approximate the derivatives. However, not only is it costly when the dimension of the parameter α is large, but it also provides biased estimators.
2. The *pathwise method* (proposed in our context by Yang and Kushner [YK91]), which consists in putting the gradient inside the expectation, involving ∇f and $\nabla_\alpha X_T$. Then, $\nabla_\alpha J(\alpha)$ is expressed as an expectation (see Proposition 1.1 below) and Monte Carlo methods can be used. One limitation of this method is that the cost function f has to be smooth.
3. The so-called *likelihood method* or *score method* (introduced by Glynn [Gly86, Gly87], Reiman and Weiss [RW86]; see also Broadie and Glasserman [BG96] for applications to the computation of Greeks in finance), in which the gradient is rewritten as $\mathbb{E}(f(X_T)H)$ for some random variable H. There is no uniqueness in this representation, since we can add to H any random variables orthogonal to X_T. Unlike the pathwise method, this method is not limited to smooth cost functions. Usually, H is equal to $\nabla_\alpha(\log(p(\alpha,X_T)))$, where $p(\alpha,.)$ is the density w.r.t. the Lebesgue measure of the law of X_T. This has some strong limitations in our context since this quantity is generally unknown. However, Yang and Kushner [YK91] provide explicit weights H, under the restrictions that α concerns only b (and not σ_j) and that the diffusion coefficient is elliptic, using the Girsanov theorem (see Proposition 2.6).

A **first purpose** of this work is to handle more general situations where both coefficients defining the SDE (1.1) depend on α. To address this issue, we provide three new approaches to express the sensitivity of $J(\alpha)$ with respect to α.

1. The first one is an extension of the likelihood approach method to the case of diffusion coefficients depending on α. It uses a direct integration-by-parts formula of the Malliavin calculus. This idea has been used recently in a financial context in the paper by Fournié et al. [FLL$^+$99] to compute option prices' sensitivities. These techniques have also been used efficiently by the first author to derive asymptotic properties of statistical procedures when we estimate parameters defining a SDE (see [Gob01b, Gob02]). Actually, our true contribution concerns essentially a situation where ellipticity is replaced by a weaker (but standard) nondegeneracy condition, which addresses random mechanics problems or portfolio optimization problems in finance.
2. The second approach is rather different from previous methods. Indeed, we initially focus on the adjoint point of view (see Bensoussan [Ben88] or Peng [Pen90]) to finally derive new formulae, involving again some integration-by-parts formula, but written in a simple way (using only Itô's calculus). In stochastic control problems, adjoint processes are related to backward SDEs (see Yong and Zhou [YZ99], e.g.), and their simulation is an extremely difficult and costly task. Here, we circumvent this difficulty since we only need to express them as explicit conditional expectations, which is feasible.
3. The third approach follows from martingale arguments applied to the expected cost and leads to an original representation, which appears to be surprisingly simple.

To compare these new methods with the previous ones, we will measure in section 5, on the one hand, the variance of the random variables involved in the resulting formulae for $\nabla_\alpha J(\alpha)$, and on the other hand, the computational time. Surprisingly, the three methods that we propose behave similarly in terms of variance, but the most efficient in terms of computational time is certainly the *martingale approach* (see Tables 5.1, 5.2, 5.3, 5.4, and 5.5).

Another element of comparison is the influence of the time step h, which is used to approximately simulate the random variables. The analysis of these discretization errors is the second significant part of this work. The relevant random variables are essentially written as the product of the cost function $f(X_T)$ by a random variable H, and simulations are based on Euler schemes. Although H has a complex form, we first propose an approximation algorithm and then we analyze the induced error w.r.t. the time step h. This part of the paper is original: previous results in the literature concern the approximation of $\mathbb{E}(f(X_T))$ (see Bally and Talay [BT96a]) or more generally of some smooth functionals of X (see Kohatsu-Higa and Pettersson [KHP00], [KHP02]). Here, regarding the techniques, we improve estimates given in [KHP00] since we do not need to add a small perturbation to the processes. Our multidimensional framework also raises extra difficulties compared to [KHP02], and we develop specific localization techniques that are interesting for themselves.

Outline of the paper. In the following, we make some assumptions and define the notations which will be used throughout the paper. We also recall the *pathwise* approach in Proposition 1.1. In section 2, after giving some standard facts on the Malliavin calculus, we develop our three approaches to computing the sensitivity of $J(\alpha)$ w.r.t. α: these are the so-called *Malliavin calculus* approach (Propositions 2.5 and 2.8), the *adjoint* approach (Theorem 2.11), and the *martingale* approach (Theorem 2.12). In section 3, we provide simulation procedures to compute $\nabla_\alpha J(\alpha)$ by the usual Monte Carlo approach using the methods developed before and analyze the influence of the time step h used for Euler-type schemes. A significant part of the paper covers these analyses which have never been developed before in the literature. The approximation results are stated in Theorems 3.1, 3.2, 3.4, and 3.5, while their proofs are postponed to section 4. Finally, numerical experiments in section 5 illustrate the developed methods: we compare the computational time, the complexity, the variance, and the time discretization error of the estimators on many examples borrowed from finance and control.

Assumptions. In our applications, the parameter is a priori multidimensional, but since in the following we will look at sensitivities w.r.t. α coordinatewise, it is not a restriction to assume that the parameter space $\mathcal{A}$ is a subset of $\mathbb{R}$ ($m = 1$).

The process defined in (1.1) depends on the parameter α, but we deliberately omit this dependence in the notation. Furthermore, the initial condition $X_0 = x$ is fixed throughout the paper. We note σ_j, the jth column vector of σ.

To study the sensitivity of J (defined in (1.2)) w.r.t. α, we may assume that coefficients are smooth enough: in what follows, k is an integer greater than 2.

ASSUMPTION (R_k). *The functions b and σ are of class C^1 w.r.t. the variables t, x, α, and for some $\eta > 0$, the following Hölder continuity condition holds:*

$$\sup_{(t,x,\alpha,\alpha')\in[0,T]\times\mathbb{R}^d\times\mathcal{A}\times\mathcal{A}} \frac{|g(t,x,\alpha) - g(t,x,\alpha')|}{|\alpha-\alpha'|^{\eta}} < \infty$$

for $g = \partial_\alpha b$ and $g = \partial_\alpha \sigma$. Furthermore, for any $\alpha \in \mathcal{A}$, the functions $b(\cdot,\cdot,\alpha)$, $\sigma(\cdot,\cdot,\alpha)$, $\partial_\alpha b(\cdot,\cdot,\alpha)$, and $\partial_\alpha\sigma(\cdot,\cdot,\alpha)$ are of class $C^{\lfloor k/2\rfloor,k}$ w.r.t. (t,x); the functions $\partial_\alpha b$ and $\partial_\alpha\sigma$ are uniformly bounded in (t,x,α), and the derivatives of b, σ, $\partial_\alpha b$, and $\partial_\alpha\sigma$ w.r.t. (t,x) are uniformly bounded as well.

Note that b and σ may be unbounded. We do not assert that the assumption above is the weakest possible, but it is sufficient for our purpose. At several places, the diffusion coefficient will be required to be uniformly elliptic, in the following sense.

ASSUMPTION (E). *σ is a squared matrix ($q = d$) such that the matrix $\sigma\sigma^*$ satisfies a uniform ellipticity condition:*

$$\forall (t,x) \in [0,T]\times\mathbb{R}^d, \qquad [\sigma\sigma^*](t,x,\alpha) \geq \mu_{\min}\,\mathrm{I}_d$$

for a real number $\mu_{\min} > 0$.

Notation.

- *Sensitivity estimators.* To clarify the connection between our methods and the estimators H which are derived, we will write $H_T^{Path.}$ for the pathwise approach (Proposition 1.1), $H_T^{Mall.Ell.}$ (resp., $H_T^{Mall.Gen.}$) for the Malliavin calculus approach in the elliptic case (resp., in the general case) (Propositions 2.5 and 2.8), $H_T^{b,Adj.}$ and $H_T^{\sigma,Adj.}$ for the adjoint approach (Theorem 2.11), and $H_T^{Mart.}$ for the martingale approach (Theorem 2.12). The subscript T refers to the time in the expected cost (1.2). Their approximations using some discretization procedure with N time steps will be denoted $H_T^{Path.,N}$, $H_T^{Mall.Ell.,N}$, and so on.
- *Differentiation.* As usual, derivatives w.r.t. α will be simply denoted with a dot, for instance, $\partial_\alpha J = \dot{J}$. If no ambiguity is possible, we will omit to write explicitly the parameter α in b, $\sigma_j \cdots$. We adopt the following usual convention on the gradients: if $\psi : \mathbb{R}^{p_2} \mapsto \mathbb{R}^{p_1}$ is a differentiable function, its gradient $\nabla_x\psi(x) = (\partial_{x_1}\psi(x), \ldots, \partial_{x_{p_2}}\psi(x))$ takes values in $\mathbb{R}^{p_1}\otimes\mathbb{R}^{p_2}$. At many places, $\nabla_x\psi(x)$ will simply be denoted $\psi'(x)$.

- *Linear algebra.* The rth column of a matrix A will be denoted A_r (or $A_{r,t}$ if A is a time dependent matrix), and the rth element of a vector a will be denoted a_r (or $a_{r,t}$ if a is a time dependent vector). A^* stands for the transpose of A. For a matrix A, the matrix obtained by keeping only the last r rows (resp., the last r columns) will be denoted $\Pi_r^R(A)$ (resp., $\Pi_r^C(A)$). For $i \in \{1, \dots, d\}$, we set $e^i = (0 \cdots 0\ 1\ 0 \cdots 0)^*$, where 1 is the ith coordinate.
- *Constants.* We will keep the same notation $K(T)$ for all finite, nonnegative, and nondecreasing functions: they do not depend on x, the function f, or further discretization steps h, but they may depend on the coefficients $b(\cdot)$ and $\sigma(\cdot)$. The generic notation $K(x,T)$ stands for any function bounded by $K(T)(1+|x|^Q)$ for $Q \geq 0$.

When a function $g(s,x,\alpha)$ is evaluated at $x = X_s^\alpha$, we may sometimes use the short notation g_s if no ambiguity is possible. For instance, (1.1) may be written as $X_t = x + \int_0^t b_s ds + \sum_{j=1}^q \int_0^t \sigma_{j,s} dW_s^j$.

Other processes related to $(X_t)_{0\leq t\leq T}$. To the diffusion X under (R_2), we may associate its flow, i.e., the Jacobian matrix $Y_t := \nabla_x X_t$, the inverse of its flow $Z_t = {Y_t}^{-1}$, and the pathwise derivative of X_t w.r.t. α, which we denote $\dot{X}_t$ (see Kunita [Kun84]). These processes solve

$$Y_t = \mathrm{I}_d + \int_0^t b'_s\ Y_s\ ds + \sum_{j=1}^q \int_0^t \sigma'_{j,s}\ Y_s\ dW_s^j, \tag{1.3}$$

$$Z_t = \mathrm{I}_d - \int_0^t Z_s(b'_s - \sum_{j=1}^q (\sigma'_{j,s})^2)\ ds - \sum_{j=1}^q \int_0^t Z_s \sigma'_{j,s}\ dW_s^j, \tag{1.4}$$

$$\dot{X}_t = \int_0^t \left(\dot{b}_s + b'_s\ \dot{X}_s\right) ds + \sum_{j=1}^q \int_0^t \left(\dot{\sigma}_{j,s} + \sigma'_{j,s}\ \dot{X}_s\right) dW_s^j. \tag{1.5}$$

Actually, since the process $(\dot{X}_t)_{0\leq t\leq T}$ satisfies a linear equation, it can also simply be written using Y_t and Z_t (apply Theorem 56 from p. 271 of Protter [Pro90]):

$$\dot{X}_t = Y_t \int_0^t Z_s \left[\left(\dot{b}_s - \sum_{j=1}^q \sigma'_{j,s} \dot{\sigma}_{j,s} \right) ds + \sum_{j=1}^q \dot{\sigma}_{j,s} dW_s^j \right]. \tag{1.6}$$

If f is continuously differentiable with an appropriate growth condition (in order to apply the Lebesgue differentiation theorem), one immediately obtains the following result (see also Yang and Kushner [YK91]), which we call the *pathwise approach*.

PROPOSITION 1.1. *Assume* (R_2). *One has* $\dot{J}(\alpha) = \mathbb{E}\big(H_T^{Path.}\big)$ *with*

$$H_T^{Path.} = f'(X_T)\dot{X}_T.$$

Hence, the gradient can still be written as an expectation, which is crucial for a Monte Carlo evaluation. One purpose of the paper is to extend this result to the case of nondifferentiable functions, by essentially writing $\dot{J}(\alpha) = \mathbb{E}\left(f(X_T)H\right)$ for some random variable H.

In what follows, we will make two types of assumption on f.

ASSUMPTION (H). f *is a bounded measurable function.*

Actually, the above boundedness property of f is not important, since in what follows, we essentially use the fact that the random variable $f(X_T)$ belongs to any $\mathbf{L}^p$. However, this assumption simplifies the analysis.

ASSUMPTION (H′). *f is a bounded measurable function and satisfies the following continuity estimate for $p_0 > 1$:*

$$\int_0^T \frac{\|f(X_T) - f(X_t)\|_{\mathbf{L}^{p_0}}}{T-t}\, dt < +\infty.$$

This $\mathbf{L}^p$-smoothness assumption of $f(X_T) - f(X_t)$ is obviously satisfied for uniformly Hölder functions with exponent β, but also for some nonsmooth functions, such as the indicator function of a domain.

PROPOSITION 1.2. *Let D be a domain of $\mathbb{R}^d$: suppose that either it has a compact and smooth boundary (say, of class C^2; see* [GT77]*), or it is a convex polyhedron ($D = \cap_{i=1}^I D_i$, where $(D_i)_{1\le i\le I}$ are half-spaces). Assume* (E), (R$_2$)*, and bounded coefficients b and σ. Then, the function $f = \mathbf{1}_D$ satisfies the assumption* (H′) *(for any $p_0 > 1$).*

Proof. Since $\|f(X_T) - f(X_t)\|^p_{\mathbf{L}^p} \le \mathbb{E}|\mathbf{1}_D(X_T) - \mathbf{1}_D(X_t)| \le \mathbb{P}(X_T \in D, X_t \notin D) + \mathbb{P}(X_T \notin D, X_t \in D)$, we only need to prove that $\mathbb{P}(X_T \in D, X_t \notin D) \le K(T)(T-t)^\beta$ with $\beta > 0$. Now, recall the standard exponential inequality $\mathbb{P}(\|X_u - x\| \ge \delta) \le K(T)\exp(-c\frac{\delta^2}{u})$ (with $c > 0$) available for $u \in]0,T]$ and $\delta \ge 0$ (see, e.g., Lemma 4.1 in [Gob00]). Combining this with the Markov property, it follows that $\mathbb{P}(X_T \in D, X_t \notin D) \le K(T)\mathbb{E}(\mathbf{1}_{X_t\notin D}\exp(-c\frac{d^2(X_t,D^c)}{(T-t)}))$. Then, a direct estimation of the above expectation using in particular a Gaussian upper bound for the density of the law of X_t (see Friedman [Fri64]) yields easily the required estimate with $\beta = \frac{1}{2}$ (see Lemma 2.8 in [Gob01a] for details). □

2. Sensitivity formulae.

In this section, we present three different approaches to evaluate $\dot{J}(\alpha)$. Before this, we introduce the Malliavin calculus material necessary to our computations.

2.1. Some basic results on the Malliavin calculus.

The reader may refer to Nualart [Nua95] (section 2.2 for the case of diffusion processes) for a detailed exposition of this section.

Put $\mathcal{H} = \mathbf{L}^2([0,T],\mathbb{R}^q)$: we will consider elements of $\mathcal{H}$ written as a row vector. For $h(.) \in \mathcal{H}$, denote by $W(h)$ the Wiener stochastic integral $\int_0^T h(t)\, dW_t$.

Let $\mathcal{S}$ denote the class of random variables of the form $F = f(W(h_1),\dots,W(h_N))$, where f is a C^∞-function with derivatives having a polynomial growth, $(h_1,\dots,h_N) \in \mathcal{H}^N$ and $N \ge 1$. For $F \in \mathcal{S}$, we define $\mathcal{D}F = (\mathcal{D}_tF := (\mathcal{D}_t^1F,\dots,\mathcal{D}_t^qF))_{t\in[0,T]}$, its derivative, as the $\mathcal{H}$-valued random variable given by $\mathcal{D}_tF = \sum_{i=1}^N \partial_{x_i} f(W(h_1),\dots,W(h_N))\ \ h_i(t)$. The operator $\mathcal{D}$ is closable as an operator from $\mathbf{L}^p(\Omega)$ to $\mathbf{L}^p(\Omega,\mathcal{H})$, for any $p \ge 1$. Its domain is denoted by $\mathbb{D}^{1,p}$ w.r.t. the norm $\|F\|_{1,p} = [\mathbb{E}|F|^p + \mathbb{E}(\|\mathcal{D}F\|^p_{\mathcal{H}})]^{1/p}$. We can define the iteration of the operator $\mathcal{D}$ in such a way that for a smooth random variable F, the derivative $\mathcal{D}^kF$ is a random variable with values on $\mathcal{H}^{\otimes k}$. As in the case $k = 1$, the operator $\mathcal{D}^k$ is closable from $S \subset \mathbf{L}^p(\Omega)$ into $\mathbf{L}^p(\Omega;\mathcal{H}^{\otimes k})$, $p \ge 1$. If we define the norm $\|F\|_{k,p} = [\mathbb{E}|F|^p + \sum_{j=1}^k \mathbb{E}(\|\mathcal{D}^jF\|^p_{\mathcal{H}^{\otimes j}})]^{1/p}$, we denote its domain by $\mathbb{D}^{k,p}$. Finally, set $\mathbb{D}^{k,\infty} = \cap_{p\ge1}\mathbb{D}^{k,p}$ and $\mathbb{D}^\infty = \cap_{k,p\ge1}\mathbb{D}^{k,p}$. One has the following chain rule property.

PROPOSITION 2.1. *Fix $p \ge 1$. For $f \in C_b^1(\mathbb{R}^d,\mathbb{R})$ and $F = (F_1,\dots,F_d)^*$ a random vector whose components belong to $\mathbb{D}^{1,p}$, $f(F) \in \mathbb{D}^{1,p}$ and for $t \ge 0$, one has*

$\mathcal{D}_t(f(F)) = f'(F)\mathcal{D}_t F$, *with the notation*

$$\mathcal{D}_t F = \begin{pmatrix} \mathcal{D}_t F_1 \\ \vdots \\ \mathcal{D}_t F_d \end{pmatrix} \in \mathbb{R}^d \otimes \mathbb{R}^q.$$

We now introduce δ, the Skorohod integral, defined as the adjoint operator of $\mathcal{D}$.

DEFINITION 2.2. *δ is a linear operator on $\mathbf{L}^2([0,T]\times\Omega,\mathbb{R}^q)$ with values in $\mathbf{L}^2(\Omega)$ such that*

1. *the domain of δ (denoted by $\mathrm{Dom}(\delta)$) is the set of processes $u \in \mathbf{L}^2([0,T]\times\Omega,\mathbb{R}^q)$ such that $\left|\mathbb{E}\big(\int_0^T \mathcal{D}_t F\cdot u_t\, dt\big)\right| \le c(u)\,\|F\|_{\mathbf{L}^2}$ for any $F\in\mathbb{D}^{1,2}$.*
2. *if u belongs to $\mathrm{Dom}(\delta)$, then $\delta(u)$ is the element of $\mathbf{L}^2(\Omega)$ characterized by the integration-by-parts formula*

$$\forall F\in\mathbb{D}^{1,2}, \qquad \mathbb{E}\big(F\,\delta(u)\big) = \mathbb{E}\left(\int_0^T \mathcal{D}_t F\cdot u_t\, dt\right). \tag{2.1}$$

In the following proposition, we outline a few properties of the Skorohod integral.

PROPOSITION 2.3.

1. *The space of weakly differentiable $\mathcal{H}$-valued variables $\mathbb{D}^{1,2}(\mathcal{H})$ belongs to $\mathrm{Dom}(\delta)$.*
2. *If u is an adapted process belonging to $\mathbf{L}^2([0,T]\times\Omega,\mathbb{R}^q)$, then the Skorohod integral and the Itô integral coincide: $\delta(u) = \int_0^T u_t\, dW_t$.*
3. *If F belongs to $\mathbb{D}^{1,2}$, then for any $u\in\mathrm{Dom}(\delta)$ such that $\mathbb{E}(F^2\int_0^T \|u_t\|^2\, dt) < +\infty$, one has*

$$\delta(F\, u) = F\,\delta(u) - \int_0^T \mathcal{D}_t F\cdot u_t\, dt, \tag{2.2}$$

whenever the right-hand side above belongs to $\mathbf{L}^2(\Omega)$.

Concerning the solution of SDEs, it is well known that under (R_k) $(k\ge 2)$ for any $t\ge 0$, the random variable X_t (resp., Y_t, Z_t, and $\dot{X}_t$) belongs to $\mathbb{D}^{k,\infty}$ (resp., $\mathbb{D}^{k-1,\infty}$). Furthermore, one has the following estimates: $\mathbb{E}\left(\sup_{0\le t\le T}\|\mathcal{D}_{r_1,\dots,r_{k'}}U_t\|^p\right)\le K(T,x)$ for $U_t = X_t$ with $1\le k'\le k$ or $U_t = Y_t, Z_t, \dot{X}_t$ with $1\le k'\le k-1$. Besides, $\mathcal{D}_s X_t$ is given by

$$\mathcal{D}_s X_t = Y_t\, Z_s\sigma(s,X_s)\,\mathbf{1}_{s\le t}. \tag{2.3}$$

Finally, we recall some standard results related to the integration-by-parts formulae. The Malliavin covariance matrix of a smooth random variable F is defined by

$$\gamma^F = \int_0^T \mathcal{D}_t F\,[\mathcal{D}_t F]^*\, dt. \tag{2.4}$$

PROPOSITION 2.4. *Let $\bar{\gamma}$ be a multi-index, F be a random variable in $\mathbb{D}^{k_1,\infty}$ such that $\det(\gamma^F)$ is almost surely positive with $1/\det(\gamma^F)\in\cap_{p\ge 1}\mathbf{L}^p$ and G belongs to $\mathbb{D}^{k_2,\infty}$. Then for any smooth function g with polynomial growth, provided that k_1 and k_2 are large enough (depending on $\bar{\gamma}$), there exists a random variable $H_{\bar{\gamma}}(F,G)$ in any $\mathbf{L}^p$ such that*

$$\mathbb{E}[\partial^{\bar{\gamma}} g(F) G] = \mathbb{E}[g(F) H_{\bar{\gamma}}(F,G)].$$

Moreover, for any arbitrary event A we have

$$\|H_{\bar\gamma}(F,G)\mathbf{1}_A\|_{\mathbf{L}^p} \le C\|[\gamma^F]^{-1}\mathbf{1}_A\|_{\mathbf{L}^{q_3}}^{p_3}\|F\|_{k_1,q_1}^{p_1}\|G\|_{k_2,q_2}$$

for some constants $C, p_1, p_3, q_1, q_2, q_3$ depending on p and $\bar\gamma$.

Proof. See Propositions 3.2.1 and 3.2.2 in Nualart [Nua98, pp. 160–161] when $A = \Omega$. For any other event A, see Proposition 2.4 from Bally and Talay [BT96a]. □

The construction of $H_{\bar\gamma}(F,G)$ is based on the equality (2.1) and involves iterated Skorohod integrals. We do not really need to make it explicit at this stage.

2.2. First approach: Direct Malliavin calculus computations. Here, the guiding idea is to start from Proposition 1.1 and apply results like Proposition 2.4 to get $\dot J(\alpha) = \mathbb{E}(f(X_T)H)$. Nevertheless, there are several ways to do this, depending on whether the diffusion coefficient is elliptic (see also [FLL$^+$99] in that situation) or not.

2.2.1. Elliptic case. Consider first that the assumption (E) is fulfilled.

PROPOSITION 2.5. *Assume* (R_2), (E)*, and* (H)*. One has $\dot J(\alpha) = \mathbb{E}\big(H_T^{Mall.Ell.}\big)$, where*

$$H_T^{Mall.Ell.} = \frac{1}{T} f(X_T)\delta\big([\sigma_\cdot^{-1}\, Y_\cdot\, Z_T\, \dot X_T]^*\big)$$

belongs to $\cap_{p\ge1}\mathbf{L}^p$.

Proof. We can consider that f is smooth, the general case being obtained using an $\mathbf{L}^2$-approximation of f with some smooth and compactly supported functions. As a consequence of (2.3) and Assumption (E), $\mathcal{D}_t X_T$ is invertible for any $t \in [0,T]$: thus, for such t, using the chain rule (Proposition 2.1), one gets that $f'(X_T) = \mathcal{D}_t(f(X_T))\sigma_t^{-1}\, Y_t\, Z_T$. Integrating in time over $[0,T]$ and using Proposition 1.1, one gets that $\dot J(\alpha) = \frac{1}{T}\int_0^T dt\, \mathbb{E}(\mathcal{D}_t(f(X_T))\sigma_t^{-1}\, Y_t\, Z_T\, \dot X_T)$. An application of the relation (2.1) completes the proof of Proposition 2.5 (the $\mathbf{L}^p$-estimates follow from Proposition 2.4). □

When the parameter enters the drift coefficient only, the laws of $(X_t)_{0\le t\le T}$ for two different values of α are equivalent owing to the Girsanov theorem. Exploiting this possible change of measure directly, a simplified expression for $\dot J(\alpha)$ can be found: this is the *likelihood ratio method* or *score method* from Kushner and Yang [YK91].

PROPOSITION 2.6. *Assume* (R_2), (E)*, and* (H)*. Suppose that the parameter of interest α is not in the diffusion coefficient. Then, one has*

$$\dot J(\alpha) = \mathbb{E}\left(f(X_T)\int_0^T [\sigma_t^{-1}\dot b_t]^* dW_t\right).$$

Proof. Instead of using the Girsanov theorem, we leverage the particular form of $\dot X_T$ given in (1.6) to prove this. Indeed, $f'(X_T)\dot X_T = f'(X_T)Y_T\int_0^T Z_t\dot b_t dt = \int_0^T dt\, \mathcal{D}_t(f(X_T))[\sigma_t^{-1}\dot b_t]$, and the result follows using (2.1). □

2.2.2. General nondegenerate case. There are many situations where the ellipticity Assumption (E) is too stringent and cannot be fulfilled. To illustrate this, let us rewrite the SDE in the following way, splitting its structure into two parts:

$$dX_t = \begin{pmatrix} dS_t \\ dV_t \end{pmatrix} = \begin{pmatrix} b_S(t,X_t,\alpha) \\ b_V(t,X_t,\alpha) \end{pmatrix} dt + \begin{pmatrix} \sigma_S(t,X_t,\alpha) \\ \sigma_V(t,X_t,\alpha) \end{pmatrix} dW_t. \tag{2.5}$$

Here, $(S_t)_{t\geq 0}$ is $(d-r)$-dimensional, $(V_t)_{t\geq 0}$ r-dimensional, and the dimension of W is arbitrary. The cost function of interest may involve only the value of V_T: $J(\alpha)=\mathbb{E}(f(V_T))$. Note that considering $r=d$ reduces to the previous situation. We now give two examples that motivate the statement of Proposition 2.7 below.

(a) In random mechanics (see Krée and Soize [KS86]), the pair position/velocity $dX_t = \begin{pmatrix} dx_t \\ dv_t \end{pmatrix} = \begin{pmatrix} v_t dt \\ \cdots \end{pmatrix}$ cannot satisfy an ellipticity condition, but weaker assumptions such as hypoellipticity are more realistic.

(b) For portfolio optimization in finance (for a recent review, see, e.g., Runggaldier [Run02]), r usually equals 1. $(S_t)_{t\geq 0}$ describes the dynamic of the risky assets, while $(V_t)_{t\geq 0}$ is the wealth process, corresponding to the value of a self-financed portfolio invested in a nonrisky asset with instantaneous return $r(t,S_t)$ and in the assets $(S_t)_{t\geq 0}$ w.r.t. the strategy $(\xi_t=\{\xi_i(t,X_t): 1\leq i\leq d-1\})_{t\geq 0}$: $dV_t=\xi(t,X_t)\cdot dS_t+(V_t-\xi(t,X_t)\cdot S_t)r(t,S_t)dt$ (see e.g. Karatzas and Shreve [KS98]). It is clear that the resulting diffusion coefficient for the whole process $X_t=\begin{pmatrix} S_t \\ V_t \end{pmatrix}$ cannot satisfy an ellipticity condition. Nevertheless, requiring that the matrix $\sigma_V\sigma_V^*(t,x)$ satisfy an ellipticity type condition is not very restricting in that framework.

We set γ_T for the Malliavin covariance matrix of V_T: $\gamma_T=\int_0^T \mathcal{D}_t V_T\ [\mathcal{D}_t V_T]^*\ dt$. This allows to reformulate Assumption (E) as the following.

Assumption (E′). $\det(\gamma_T)$ *is almost surely positive and for any* $p\geq 1$*, one has*

$$\|1/\det(\gamma_T)\|_{\mathbf{L}^p}<+\infty.$$

We now bring together standard results related to Assumption (E′).

Proposition 2.7. *Assumption* (E′) *is fulfilled in the following situations.*

1. Hypoelliptic case *(with $r=d$) under* (R_∞). *The Lie algebra generated by the vector fields* $\partial_t+A_0(t,x):=\partial_t+\sum_{i=1}^d(b-\frac{1}{2}\sum_{j=1}^q\sigma_j'\sigma_j)_i(t,x)\partial_{x_i}$, $A_j(t,x):=\sum_{i=1}^d\sigma_{i,j}(t,x)\partial_{x_i}$ *for* $1\leq j\leq q$ *spans* $\mathbb{R}^{d+1}$ *at the point* $(0,X_0)$*:*

$$\dim \operatorname{span} \operatorname{Lie}(\partial_t+A_0,A_j,1\leq j\leq q)(0,X_0)=d+1.$$

2. Partially elliptic case *(with $r\geq 1$) under* (R_2). *For a real number* $\mu_{\min}>0$*, one has*

$$\forall x\in\mathbb{R}^d,\qquad [\sigma_V\sigma_V^*](T,x,\alpha)\geq \mu_{\min}\,\mathrm{I}_d.$$

Proof. The statement 1 is standard and we refer to Cattiaux and Mesnager [CM02] for a recent account on the subject. The statement 2 is also standard: see, for instance, the arguments in Nualart [Nua98, pp. 158–159]. □

Now, we are in a position to give a sensitivity formula under (E′).

Proposition 2.8. *Assume* (R_2), (E′)*, and* (H). *One has* $\dot{J}(\alpha)=\mathbb{E}\big(H_T^{Mall.Gen.}\big)$ *where*

$$H_T^{Mall.Gen.}=f(V_T)\ \delta\big(\dot{V}_T^*\gamma_T^{-1}\mathcal{D}_.V_T\big)$$

belongs to $\cap_{p\geq 1}\mathbf{L}^p$.

Proof. Assumption (E′) validates (see Nualart [Nua98, Proposition 3.2.1]) the following computations, adapted from the ones used for Proposition 2.5. The chain rule property yields $f'(V_T) = \int_0^T \mathcal{D}_t(f(V_T))[\mathcal{D}_t V_T]^* \gamma_T^{-1}\, dt$, and thus $\mathbb{E}(f'(V_T)\dot V_T) = \mathbb{E}(\int_0^T \mathcal{D}_t(f(V_T))[\mathcal{D}_t V_T]^* \gamma_T^{-1} \dot V_T\, dt)$. Proposition 2.8 now follows from (2.1). □

Proposition 2.8 is also valid under (E) in the case $r = d$, but the formula in Proposition 2.5 is actually a bit simpler to implement.

2.3. A second approach based on the adjoint point of view.

2.3.1. Another representation of the sensitivity of $J(\alpha)$. If we set $u(t,x) = \mathbb{E}\big(f(X_T)|X_t = x\big)$, omitting to indicate the dependence w.r.t. α, we have that $J(\alpha)$ defined in (1.2) equals $u(0, X_0)$. Under smoothness assumptions on b and σ and the nondegeneracy hypothesis on the infinitesimal generator of $(X_t)_{t\geq 0}$, it is well known (see Cattiaux and Mesnager [CM02]) that u is the smooth solution of the partial differential equation (PDE)

$$\begin{cases} \partial_t u(t,x) + \sum_{i=1}^d b_i(t,x)\partial_{x_i} u(t,x) + \frac{1}{2}\sum_{i,j=1}^d [\sigma\sigma^*]_{i,j}(t,x)\partial^2_{x_i,x_j} u(t,x) = 0 \text{ for } t < T, \\ u(T,x) = f(x). \end{cases}$$

Our purpose is to give another expression for $\dot J(\alpha)$ of Proposition 1.1. The idea is simple: it consists in formally differentiating the PDE above w.r.t. α and in reinterpreting the derivative as an expectation. This is now stated and justified rigorously.

LEMMA 2.9. *Assume* (R$_3$)*,* (E)*, and* (H)*. One has*

$$\dot J(\alpha) = \int_0^T \mathbb{E}\left(\sum_{i=1}^d \dot b_{i,t}\partial_{x_i} u(t, X_t) + \frac{1}{2}\sum_{i,j=1}^d [\dot{\sigma\sigma^*}]_{i,j,t}\partial^2_{x_i,x_j} u(t, X_t)\right) dt.$$

Proof. This is a standard fact that under (R$_3$) and (E), u is twice differentiable w.r.t. x (see the arguments of Lemma 2.10 below, where the proof is sketched). The technical difficulty in the following computations comes from the possible explosion of derivatives of u for t close to T, when f is nonsmooth. For this reason, we first prove useful uniform estimates: for any multi-index $\bar\gamma$ with $|\bar\gamma| \leq 2$, any smooth random variable $G \in \mathbb{D}^{2,\infty}$ and any parameters α and α', one has

$$\sup_{t\in[0,T[} |\mathbb{E}[G\, \partial_x^{\bar\gamma} u(t, X_t^{\alpha'})]| \leq K(T,x)\frac{\|f\|_\infty}{T^{\frac{|\bar\gamma|}{2}}}\|G\|_{|\bar\gamma|,p'}. \tag{2.6}$$

Indeed, for $t \geq T/2$, first apply Proposition 2.4: then, use $|u(t,x)| \leq \|f\|_\infty$ combined with some specific estimates for $\|H_{\bar\gamma}(X_t^{\alpha'}, G)\|_{\mathbf{L}^p} \leq \frac{K(T,x)}{t^{\frac{|\bar\gamma|}{2}}}\|G\|_{|\bar\gamma|,p'}$ available under the ellipticity condition (E) (see Theorem 1.20 and Corollary 3.7 in Kusuoka and Stroock [KS84], or section 4.1. in [Gob00] for a brief review). For $t \leq T/2$, note that using the Markov property, one has $\partial_x^{\bar\gamma} u(t,x) = \partial_x^{\bar\gamma}\mathbb{E}\big(u(\frac{T+t}{2}, X^{t,x}_{\frac{T+t}{2}})\big) = \sum_{1\leq|\gamma'|\leq|\bar\gamma|}\mathbb{E}\big(\partial_x^{\gamma'} u(\frac{T+t}{2}, X^{t,x}_{\frac{T+t}{2}}) G^{\gamma'}_{\frac{T+t}{2}}\big)$ with $G^{\gamma'}_{\frac{T+t}{2}} \in \mathbb{D}^{2+|\gamma'|-|\bar\gamma|,\infty}$ and $(X_s^{t,y})_{s\geq t}$ standing for the process starting from y at time t. Again applying the integration-by-parts formula with the elliptic estimates gives $|\partial_x^{\bar\gamma} u(t,x)| \leq \frac{K(T,x)}{[\frac{T+t}{2}-t]^{\frac{|\bar\gamma|}{2}}}\|f\|_\infty$ and (2.6) follows since $\frac{T+t}{2} - t \geq \frac{T}{4}$. Now, for $\epsilon \in \mathbb{R}$, the difference $J(\alpha+\epsilon) - J(\alpha)$ equals

$$\begin{aligned}
&\mathbb{E}\left(f(X_T^{\alpha+\epsilon}) - f(X_T^{\alpha})\right) = \mathbb{E}\left(u(T, X_T^{\alpha+\epsilon}) - u(0, X_0^{\alpha+\epsilon})\right)\\
&= \int_0^T \mathbb{E}\left(\partial_t u(t, X_t^{\alpha+\epsilon}) + \sum_{i=1}^d b_i(t, X_t^{\alpha+\epsilon}, \alpha+\epsilon)\partial_{x_i} u(t, X_t^{\alpha+\epsilon})\right.\\
&\qquad\qquad \left. + \frac{1}{2}\sum_{i,j=1}^d [\sigma\sigma^*]_{i,j}(t, X_t^{\alpha+\epsilon}, \alpha+\epsilon)\partial^2_{x_i,x_j} u(t, X_t^{\alpha+\epsilon})\right) dt\\
&= \int_0^T \mathbb{E}\left(\sum_{i=1}^d (b_i(t, X_t^{\alpha+\epsilon}, \alpha+\epsilon) - b_i(t, X_t^{\alpha+\epsilon}, \alpha))\partial_{x_i} u(t, X_t^{\alpha+\epsilon})\right.\\
&\qquad\qquad \left. + \frac{1}{2}\sum_{i,j=1}^d ([\sigma\sigma^*]_{i,j}(t, X_t^{\alpha+\epsilon}, \alpha+\epsilon) - [\sigma\sigma^*]_{i,j}(t, X_t^{\alpha+\epsilon}, \alpha))\partial^2_{x_i,x_j} u(t, X_t^{\alpha+\epsilon})\right) dt,
\end{aligned}$$

where at the last equality we used the PDE solved by u to remove the term $\partial_t u$. Now, divide by ϵ and take its limit to 0: the result follows owing to the uniform estimates (2.6). ∎

Note that the formulation of Lemma 2.9 is strongly related to a form of the stochastic maximum principle (the Pontryagin principle) for optimal control problems: the processes $([\partial_{x_i} u(t, X_t)]_i)_{0 \leq t < T}$ and $([\partial^2_{x_i,x_j} u(t, X_t)]_{i,j})_{0 \leq t < T}$ are the so-called adjoint processes (see Bensoussan [Ben88] for convex control domains, or more generally Peng [Pen90]) and solve backward SDEs. Usually in these problems, the function f is smooth. Here, since the law of X_t has a smooth density w.r.t. the Lebesgue measure, we can remove the regularity condition on f.

Note also that Lemma 2.9 remains valid under a hypoellipticity hypothesis (condition 1 in Proposition 2.7). However, the derivation of tractable formulae below relies strongly on the ellipticity property.

2.3.2. Transformation using Itô–Malliavin integration-by-parts formulae. The aim of this section is to transform the expression for $\dot{J}(\alpha)$ in terms of explicit quantities. To remove the nonexplicit terms $\partial_{x_i} u$ and $\partial^2_{x_i,x_j} u$, we may use some integration-by-parts formulae, but here, to keep more tractable expressions, we are going to derive Bismut-type formulae, i.e., involving only Itô integrals instead of Skorohod integrals (see Bismut [Bis84]; Elworthy, Le Jan, and Li [EJL99]; and references therein), using a martingale argument (see also Thalmaier [Tha97] or, more recently, Picard [Pic02]). In the cited references, this approach has been used to compute estimates of the gradient of u. Here, we extend it to support higher derivatives. The basic tool is given by the following lemma.

LEMMA 2.10. *Assume* (R$_2$), (E), *and* (H) *and define* $M_t = u'(t, X_t)Y_t$ *for* $t < T$. *Then* $M = (M_t)_{0 \leq t < T}$ *is an* $\mathbb{R}^1 \otimes \mathbb{R}^d$*-valued martingale.*

Proof. First, we justify that u is continuously differentiable w.r.t x under (R$_2$) and (E). If f is smooth, this is clear (even without (E)), but (2.10) below also shows that under (E), u' can be expressed without the derivative of f. This easily leads to our assertion (see the proof of Proposition 3.2 in [FLL$^+$99]). Now, the Markov property ensures that $(u(t, X_t^{0,x}))_{0 \leq t < T}$ is a martingale for any $x \in \mathbb{R}^d$. Hence, its derivative w.r.t. x (i.e., $(M_t)_{0 \leq t < T}$) is also a martingale (see Arnaudon and Thalmaier [AT98]). ∎

We now state a theorem which, if combined with Lemma 2.9, leads to an alternative representation for $\dot{J}(\alpha)$.

THEOREM 2.11. *Assume* (R_3) *and* (E).

Under (H), *one has*

$$\int_0^T \mathbb{E}\Big(\sum_{i=1}^d \dot{b}_{i,t}\partial_{x_i} u(t,X_t)\Big)dt = \mathbb{E}(H_T^{b,Adj.}), \tag{2.7}$$

where $H_T^{b,Adj.} = f(X_T)\int_0^T dt\, \dot{b}_t \cdot \frac{Z_t^*}{T-t}\int_t^T [\sigma_s^{-1}\, Y_s]^*\, dW_s$ *belongs to* $\bigcap_{p\geq 1}\mathbf{L}^p$.

Under (H′), *one has*

$$\int_0^T \mathbb{E}\Big(\sum_{i,j=1}^d [\sigma\dot{\sigma}^*]_{i,j,t}\partial^2_{x_i,x_j} u(t,X_t)\Big)dt = \mathbb{E}(H_T^{\sigma,Adj.}), \tag{2.8}$$

where

$$H_T^{\sigma,Adj.} = \int_0^T dt \sum_{i,j=1}^d [\sigma\dot{\sigma}^*]_{i,j,t}[f(X_T)-f(X_t)]\left(\frac{2e^j}{T-t}\cdot\left[Z_t^*\int_{\frac{T+t}{2}}^T[\sigma_s^{-1}Y_s]^*dW_s\right]\right.$$
$$\left.\times\ \frac{2e^i}{T-t}\cdot\left[Z_t^*\int_t^{\frac{T+t}{2}}[\sigma_s^{-1}Y_s]^*dW_s\right] + \frac{2e^i}{T-t}\cdot\left\{\nabla_x\left[Z_t^*\int_t^{\frac{T+t}{2}}[\sigma_s^{-1}Y_s]^*dW_s\right]Z_te^j\right\}\right)$$

belongs to $\bigcap_{p<p_0}\mathbf{L}^p$.

Proof. *Equality* (2.7). First, Clark and Ocone's formula [Nua95, p. 42] gives $u(\tau,X_\tau) = u(t,X_t) + \int_t^\tau \mathbb{E}(\mathcal{D}_s[u(\tau,X_\tau)]|\mathcal{F}_s)dW_s$ for $0\le t\le\tau<T$. Using (2.3) and the martingale property of Lemma 2.10, we get $\mathbb{E}(\mathcal{D}_s[u(\tau,X_\tau)]|\mathcal{F}_s) = \mathbb{E}(u'(\tau,X_\tau)Y_\tau Z_s\sigma_s|\mathcal{F}_s) = u'(s,X_s)\sigma_s$. Hence, it gives an explicit form to the predictable representation theorem:

$$\forall\, 0\le t\le\tau\le T \qquad u(\tau,X_\tau) = u(t,X_t) + \int_t^\tau u'(s,X_s)\sigma_s dW_s \tag{2.9}$$

(the case $\tau = T$ is obtained by passing to the limit). Note that this representation holds under (R_2). Since $(u'(t,X_t)Y_t)_{0\le t<T}$ is a martingale, we obtain that

$$\begin{aligned}
u'(t,X_t)Y_t &= \mathbb{E}\left(\frac{1}{T-t}\int_t^T u'(s,X_s)Y_s\, ds|\mathcal{F}_t\right)\\
&= \mathbb{E}\left(\frac{1}{T-t}\left[\int_t^T u'(s,X_s)\sigma_s dW_s\right]\left[\int_t^T[\sigma_s^{-1}Y_s]^*dW_s\right]^*|\mathcal{F}_t\right)\\
&= \mathbb{E}\left(\frac{f(X_T)-u(t,X_t)}{T-t}\left[\int_t^T[\sigma_s^{-1}Y_s]^*dW_s\right]^*|\mathcal{F}_t\right)\\
&= \mathbb{E}\left(\frac{f(X_T)}{T-t}\left[\int_t^T[\sigma_s^{-1}Y_s]^*dW_s\right]^*|\mathcal{F}_t\right), \qquad (2.10)
\end{aligned}$$

where for the third equality we used (2.9) with $\tau = T$ and $u(T, X_T) = f(X_T)$. Now the proof of (2.7) is straightforward.

Equality (2.8). Note that a slight modification of the preceding arguments (namely, integrating over $[t, (T+t)/2]$ instead of $[t, T]$ and applying (2.9) with $\tau = (t+T)/2$) leads to $\partial_{x_i} u(t, X_t) = \mathbb{E}\big(u\big(\frac{T+t}{2}, X_{\frac{T+t}{2}}\big)\frac{2e^i}{T-t} \cdot \big[Z_t^* \int_t^{\frac{T+t}{2}} [\sigma_s^{-1} Y_s]^* dW_s\big] | \mathcal{F}_t\big)$. Differentiating w.r.t. x on both sides and using (2.10) yields

$$\begin{aligned}(\partial_{x_i} u)'(t, X_t) Y_t &= \mathbb{E}\left(u'\left(\frac{T+t}{2}, X_{\frac{T+t}{2}}\right) Y_{\frac{T+t}{2}} \frac{2e^i}{T-t} \cdot \left[Z_t^* \int_t^{\frac{T+t}{2}} [\sigma_s^{-1} Y_s]^* dW_s\right] | \mathcal{F}_t\right)\\
&\quad + \mathbb{E}\left(u\left(\frac{T+t}{2}, X_{\frac{T+t}{2}}\right) \frac{2e^i}{T-t} \cdot \nabla_x \left\{\left[Z_t^* \int_t^{\frac{T+t}{2}} [\sigma_s^{-1} Y_s]^* dW_s\right]\right\} | \mathcal{F}_t\right)\\
&= \mathbb{E}\Bigg([f(X_T) - f(X_t)] \frac{2}{T-t}\left[\int_{\frac{T+t}{2}}^T [\sigma_s^{-1} Y_s]^* dW_s\right]^* \frac{2e^i}{T-t} \cdot \left[Z_t^* \int_t^{\frac{T+t}{2}} [\sigma_s^{-1} Y_s]^* dW_s\right]\\
&\quad + [f(X_T) - f(X_t)] \frac{2e^i}{T-t} \cdot \nabla_x \left\{\left[Z_t^* \int_t^{\frac{T+t}{2}} [\sigma_s^{-1} Y_s]^* dW_s\right]\right\} | \mathcal{F}_t\Bigg)\end{aligned}$$

(note that the $f(X_t)$ terms have no contribution in the expectation). Rearranging this last expression leads to (2.8).

The $\mathbf{L}^p$-estimates can be justified using the generalized Minkowski inequality and standard estimates from the stochastic calculus:

(2.11)
$$\begin{aligned}&\|H_T^{b,Adj.}\|_{\mathbf{L}^p} \le \int_0^T \frac{\|f\|_\infty}{T-t} \left\|\dot{b}(t, X_t) \cdot Z_t^* \int_t^T [\sigma_s^{-1} Y_s]^* dW_s\right\|_{\mathbf{L}^p} dt \le K(T, x) \int_0^T \frac{\|f\|_\infty}{\sqrt{T-t}} dt,\\
&\|H_T^{\sigma,Adj.}\|_{\mathbf{L}^p} \le K(T, x) \int_0^T \frac{\|f(X_T) - f(X_t)\|_{\mathbf{L}^{p'}}}{T-t}\, dt\end{aligned}$$

for $p < p' < p_0$. □

Remark 2.1. The $f(X_t)$ terms in $H_T^{\sigma,Adj.}$ seem to be crucial to ensure its $\mathbf{L}^p$ integrability: numerical experiments in section 5 illustrate this fact.

2.4. A third approach using martingales. We emphasize the dependence on α of the expected cost by denoting $u(\alpha, t, x) = \mathbb{E}(f(X_T^\alpha) | X_t^\alpha = x)$: hence, $J(\alpha) = u(\alpha, 0, X_0)$. From the estimates proved in Lemma 2.9, this is a differentiable function w.r.t. α and one has $|\dot{u}(\alpha, t, x)| \le K(T, x)\|f\|_\infty$ and $|u'(\alpha, t, x)| \le \frac{K(T,x)}{\sqrt{T-t}}\|f\|_\infty$. Furthermore, using Theorem 2.11 and the $\mathbf{L}^p$-estimates (2.11) under (H$'$), one gets

$$|\dot{u}(\alpha, t, x)| \le K(T, x)\left[\|f\|_\infty \sqrt{T-t} + \int_t^T \frac{\|f(X_T^{t,x}) - f(X_s^{t,x})\|_{\mathbf{L}^{p'}}}{T-s}\, ds\right]$$

for $p' < p_0$. Consequently, if we put $g(r) = \mathbb{E}\left(\dot{u}(\alpha, r, X_r)\right)$, we easily obtain $|g(r)| \le K(T, x)[\|f\|_\infty \sqrt{T-r} + \int_r^T \frac{\|f(X_T) - f(X_s)\|_{\mathbf{L}^{p_0}}}{T-s}\, ds]$ and thus, $\lim_{r \to T} g(r) = 0$. For any $0 \le r \le s \le T$, one has $\mathbb{E}(u(\alpha, r, X_r)) = \mathbb{E}(u(\alpha, s, X_s)) = \frac{1}{T-r}\int_r^T \mathbb{E}(u(\alpha, s, X_s))\, ds$

using the Markov property; hence, by differentiation w.r.t. α, one gets

(2.12)
$$\begin{aligned}\mathbb{E}\big(\dot{u}(\alpha,r,X_r)\big) &= \frac{1}{T-r}\int_r^T ds\ \mathbb{E}\big(\dot{u}(\alpha,s,X_s)+u'(\alpha,s,X_s)\dot{X}_s - u'(\alpha,r,X_r)\dot{X}_r\big)\\ &= \frac{1}{T-r}\int_r^T ds\ \mathbb{E}\big(\dot{u}(\alpha,s,X_s)+u'(\alpha,s,X_s)[\dot{X}_s - Y_s Z_r \dot{X}_r]\big),\end{aligned}$$

where we used at the last equality the martingale property of $M_t = u'(\alpha,t,X_t)Y_t$ between $t=s$ and $t=r$ (see Lemma 2.10).

Now, put $h(r) = \frac{1}{T-r}\int_r^T ds\ \mathbb{E}\left(u'(\alpha,s,X_s)[\dot{X}_s - Y_s Z_r \dot{X}_r]\right)$: one has derived the following integral equation:

$$g(t) = \frac{1}{T-t}\int_t^T g(s)\ ds + h(t). \tag{2.13}$$

Before solving it, we express $h(r)$ using only f: for this, we use the predictable representation (2.9), which immediately gives

$$h(r) = \frac{1}{T-r}\mathbb{E}\left((f(X_T)-f(X_r))\int_r^T [\sigma_s^{-1}(\dot{X}_s - Y_s Z_r \dot{X}_r)]^* dW_s\right). \tag{2.14}$$

Note again that the term with $f(X_r)$ has no contribution and is put only to justify that $|h(r)| \le K(T,x)\|f(X_T)-f(X_r)\|_{\mathbf{L}^{p_0}}$ (use the Burkholder–Davis–Gundy inequalities and straightforward upper bounds for $\|\dot{X}_s - Y_s Z_r \dot{X}_r\|_{\mathbf{L}^q} \le K(T,x)\sqrt{s-r}$), from which we deduce that the integral $\int_0^T \frac{h(t)}{T-t}\,dt$ is convergent because of (H′). To solve the integral equation above, note that $[\frac{1}{T-t}\int_t^T g(s)\ ds]' = -\frac{h(t)}{T-t}$, and thus by integration, we have $\frac{1}{T-t}\int_t^T g(s)\ ds = C - \int_t^T \frac{h(r)}{T-r}dr$. The constant C equals 0 since both integrals in the previous equality converge to 0 when t goes to T (use $\lim_{t\to T} g(t) = 0$ and (H′)). Plug this new equality into (2.13), use (2.14), and take $t=0$ (with $\dot{X}_0 = 0$) to get the following representation for $\dot{J}(\alpha)$: this is the main result of this section.

THEOREM 2.12. *Assume* (R_2)*,* (E)*, and* (H′)*. Then, one has* $\dot{J}(\alpha) = \mathbb{E}(H_T^{Mart.})$ *with*

$$\begin{aligned}H_T^{Mart.} &= \frac{f(X_T)}{T}\int_0^T [\sigma_s^{-1}\dot{X}_s]^* dW_s\\ &\quad + \int_0^T dr\ \frac{[f(X_T)-f(X_r)]}{(T-r)^2}\int_r^T [\sigma_s^{-1}(\dot{X}_s - Y_s Z_r \dot{X}_r)]^* dW_s.\end{aligned} \tag{2.15}$$

Furthermore, the random variable $H_T^{Mart.}$ *belongs to* $\bigcap_{p<p_0}\mathbf{L}^p$*.*

This method is called the *martingale approach* because it is based on the equality (2.12), which is a consequence of the martingale property of

$$[\dot{u}(\alpha,s,X_s) + u'(\alpha,s,X_s)\dot{X}_s]_{0\le s<T}.$$

Proof. What remains to be proved is the $\mathbf{L}^p$ estimate of $H_T^{Mart.}$: this can be easily obtained by combining Minkowski's inequality, Hölder's inequality, Assumption (H′), and standard stochastic calculus inequalities as before. ∎

Remark 2.2. When the parameter is not involved in the diffusion coefficient, it is easy to see that the improved estimate $\|\dot{X}_s - Y_s Z_r \dot{X}_r\|_{L_q} \leq K(T,x)(s-r)$ is available: thus, this allows us to remove $f(X_r)$ terms in the expression of $H_T^{Mart.}$ without changing the finiteness of the $\mathbf{L}^p$-norm of the new $H_T^{Mart.}$. In other words, only Assumption (H) is needed.

Besides, when α is only in the drift coefficient and these $f(X_r)$ terms are suppressed, this representation coincides with that of Theorem 2.11. Indeed, let us write $P_r = \int_r^T [\sigma_s^{-1}(\dot{X}_s - Y_s Z_r \dot{X}_r)]^* dW_s = \int_r^T [\sigma_s^{-1}\dot{X}_s]^* dW_s - [Z_r\dot{X}_r]^* \int_r^T [\sigma_s^{-1} Y_s]^* dW_s :=$ $P_{1,r} - P_{2,r}$, where $P_{1,r} = \int_0^T [\sigma_s^{-1}\dot{X}_s]^* dW_s - [Z_r\dot{X}_r]^* \int_0^T [\sigma_s^{-1} Y_s]^* dW_s$ and $P_{2,r} =$ $\int_0^r [\sigma_s^{-1}\dot{X}_s]^* dW_s - [Z_r\dot{X}_r]^* \int_0^r [\sigma_s^{-1} Y_s]^* dW_s$. From the fact that $Z_r\dot{X}_r = \int_0^r Z_s \dot{b}_s ds$ (see (1.6)), one gets $dP_{2,r} = [Z_r\dot{b}_r]^* (\int_0^r [\sigma_t^{-1} Y_t]^* dW_t) dr$, hence $P_{2,r}$ is of bounded variation. $P_{1,r}$ is also of bounded variation, since $Z_r\dot{X}_r$ is. Thus, one obtains $dP_r = -\dot{b}_r \cdot Z_r^* (\int_r^T [\sigma_t^{-1} Y_t]^* dW_t) dr$: furthermore, since $P_T = 0$, one has $\|P_r\|_{\mathbf{L}^p} \leq$ $K(T,x)(T-r)^{3/2}$. Using an integration-by-parts formula in (2.15) finally completes our assertion: $H_T^{Mart.} = f(X_T)\big(\frac{1}{T}P_0 + \int_0^T \frac{P_r}{(T-r)^2} dr\big) = f(X_T)\big(-\int_0^T \frac{dP_r}{(T-r)}\big) = H_T^{b,Adj.}$.

Consequently, this martingale approach does not provide any new elements when the parameter is not in the diffusion coefficient. On the contrary, if σ depends on α, the representation with the adjoint point of view is different from the martingale one (see numerical experiments). However, we must admit that this martingale approach remains somewhat mysterious to us.

3. Monte Carlo simulation and analysis of the discretization error. In this section, we discuss the numerical implementation of the formulae derived in this paper to compute the sensitivity of $J(\alpha)$ w.r.t. α. These formulae are written as expectations of some functionals of the process $(X_t)_{0\leq t\leq T}$ and related ones: a standard way to proceed consists in drawing independent simulations, approximating the functional using Euler schemes, and averaging independent samples of the resulting functional to get an estimation of the expectation (see section 5).

Here, we focus on the impact of the time step $h = T/N$ (N is the number of discretization times in the regular mesh of the interval $[0,T]$) in the simulation of the functional: it is well known that for the evaluation of $\mathbb{E}(f(X_T))$, the discretization error using an Euler scheme is of order h (see Bally and Talay [BT96a] for measurable functions f, or Kohatsu-Higa and Pettersson [KHP02] if f is a distribution and for more general discretization schemes). We recall that the error on the processes (called the *strong* error) is much easier to analyze than the one on the expectations (the *weak* error): the first one is essentially of order $\sqrt{h}$ (see [KP95]) but this is not relevant for the current issues.

Besides, the quantity of interest here has a more complex structure that is essentially $\mathbb{E}(f(X_T)H)$, where H is one of the random variables resulting from our computations. In general, H involves Itô or Skorohod integrals: our first purpose is to give some approximation procedure to simulate these weights using only the increments of the Brownian motion computed along the regular mesh with time step h.

Our second purpose is to analyze the error induced by this discretization procedure: generally speaking, the weak error is still at most linear w.r.t. h, as for $\mathbb{E}(f(X_T))$. The proofs are quite intricate and we postpone them to section 4. For the sake of clarity, we assume (R_∞), that is, b and σ of class C^∞, but approximation results only depend on a finite number of coefficients' derivatives.

Approximation procedure. We consider a regular mesh of the interval $[0,T]$, with N discretization times $t_i = ih$, where $h = T/N$ is the time step. Denote $\phi(t) = \sup\{t_i : t_i \le t\}$. The processes we need to simulate are essentially $(X_t)_{0\le t\le T}$, $(Y_t)_{0\le t\le T}$, $(Z_t)_{0\le t\le T}$, $(\dot X_t)_{0\le t\le T}$, and we approximate them using a standard Euler scheme as follows:

$$X_t^N = x + \int_0^t b(\phi(s), X^N_{\phi(s)})\, ds + \sum_{j=1}^q \int_0^t \sigma_j(\phi(s), X^N_{\phi(s)})\, dW_s^j, \tag{3.1}$$

$$Y_t^N = \mathrm{I}_d + \int_0^t b'(\phi(s), X^N_{\phi(s)})\, Y^N_{\phi(s)}\, ds + \sum_{j=1}^q \int_0^t \sigma'_j(\phi(s), X^N_{\phi(s)})\, Y^N_{\phi(s)}\, dW_s^j, \tag{3.2}$$

$$Z_t^N = \mathrm{I}_d - \int_0^t Z^N_{\phi(s)}(b' - \sum_{j=1}^q (\sigma'_j)^2)(\phi(s), X^N_{\phi(s)})\, ds - \sum_{j=1}^q \int_0^t Z^N_{\phi(s)}\sigma'_j(\phi(s), X^N_{\phi(s)})\, dW_s^j, \tag{3.3}$$

$$\begin{aligned}\dot X_t^N = &\int_0^t \left(\dot b(\phi(s), X^N_{\phi(s)}) + b'(\phi(s), X^N_{\phi(s)})\, \dot X^N_{\phi(s)}\right)\, ds\\ &+\sum_{j=1}^q \int_0^t \left(\dot\sigma_j(\phi(s), X^N_{\phi(s)}) + \sigma'_j(\phi(s), X^N_{\phi(s)})\, \dot X^N_{\phi(s)}\right)\ dW_s^j.\end{aligned} \tag{3.4}$$

Note that only the increments $(W^j_{t_{i+1}} - W^j_{t_i}; 1 \le j \le q)_{0\le i\le N-1}$ of the Brownian motion are needed to get values of X^N, Z^N, Y^N, $\dot X^N$ at times $(t_i)_{0\le i\le N}$.

3.1. Pathwise approach.

THEOREM 3.1. *Assume* (R_∞). *Then, one has*

$$\left|\dot J(\alpha) - \mathbb{E}\left(f'(X_T^N)\dot X_T^N\right)\right| \le C(T,x,f)h,$$

under either one of the two following assumptions on f and X:

(A1) *f is of class C_b^4: one may put $C(T,x,f) = K(T,x)\sum_{1\le|\alpha|\le 4} \|\partial^\alpha f\|_\infty$ in that case.*

(A2) *f is continuously differentiable with a bounded gradient and the nondegeneracy condition* (E') *holds: in that case, $C(T,x,f)$ may be set to*

$$K(T,x)\|f'\|_\infty \|1/\det(\gamma_T)\|^q_{\mathbf{L}^p}$$

for some positive numbers p and q.

Note that in the case (A1), only three additional derivatives of the function f' are required to get the order 1 w.r.t. h: this is a slight improvement compared to results in Talay and Tubaro [TL90], where four derivatives are needed.

3.2. Malliavin calculus approach.

3.2.1. Elliptic case. One needs to define the approximation for the random variable $H_T^{Mall.Ell.} := \delta\big([\sigma^{-1}(\cdot, X.)\ Y.\ Z_T\ \dot X_T]^*\big)$ involved in Proposition 2.5. Basic algebra using the equality (2.2) gives

$$\begin{aligned}H_T^{Mall.Ell.} &= \sum_{i=1}^d \delta\big([\sigma^{-1}(\cdot,X_\cdot)\,Y_\cdot]_i^*\,[Z_T\,\dot X_T]_i\big)\\ &= \sum_{i=1}^d [Z_T\,\dot X_T]_i \int_0^T [\sigma^{-1}(s,X_s)\,Y_s]_i^*\,dW_s - \sum_{i=1}^d \int_0^T \mathcal{D}_s([Z_T\,\dot X_T]_i)[\sigma^{-1}(s,X_s)\,Y_s]_i\,ds.\end{aligned}$$

The new quantities involved are $\mathcal{D}_s Z_{j,k,T}$ and $\mathcal{D}_s \dot X_{k,T}$. We now indicate how to simulate them. The $\mathbb{R}^{2d}$-valued process $\binom{X_t}{\dot X_t}_{t\geq 0}$ forms a new stochastic differential equation (see (1.5)): we denote the flow of this extended system by $\hat Y_t$ and its inverse by $\hat Z_t$. As we did for Y_t and Z_t, we can define their Euler scheme (as in (3.2) and (3.3)), which we denote $\hat Y_t^N$ and $\hat Z_t^N$. The Malliavin derivative of this system follows from (2.3). Hence, one has

$$\mathcal{D}_s \dot X_T = \Pi_d^R \left(\hat Y_T \hat Z_s \begin{pmatrix} \vdots & \sigma_j(s,X_s) & \vdots \\ \vdots & \dot\sigma_j(s,X_s) + \sigma_j'(s,X_s)\,\dot X_s & \vdots \end{pmatrix}\right), \tag{3.5}$$

and we naturally approximate it by

$$[\mathcal{D}_s \dot X_T]^N = \Pi_d^R \left(\hat Y_T^N \hat Z_s^N \begin{pmatrix} \vdots & \sigma_j(s,X_s^N) & \vdots \\ \vdots & \dot\sigma_j(s,X_s^N) + \sigma_j'(s,X_s^N)\,\dot X_s^N & \vdots \end{pmatrix}\right). \tag{3.6}$$

The same approach can be developed for the cth column of the transpose of Z_T, since $\binom{X_t}{(Z_t{}^*)_c}_{t\geq 0}$ forms a new SDE (see (1.4)): the associated flow and its inverse, respectively denoted $\hat Y_t^c$ and $\hat Z_t^c$, enable us to derive a simple expression for $\mathcal{D}_s[(Z_t{}^*)_c]$ analogously to (3.5) and (3.6). As a consequence, one gets

$$\mathcal{D}_s([Z_T\,\dot X_T]_i) = \mathbf{1}_{s\leq T} \sum_j A_{\beta(j,i),T} B_{\beta(j,i),s}, \tag{3.7}$$

where $A_{\beta(j,i),T}$ and $B_{\beta(j,i),s}$ are given by some appropriate coordinates of the processes $\hat Y_T$, $(\hat Y_T^c)_{1\leq c\leq d}$ on one hand; and $\hat Z_s$, $(\hat Z_s^c)_{1\leq c\leq d}$, $\sigma_j(s,X_s)$, $\dot\sigma_j(s,X_s)$, $\sigma_j'(s,X_s)$, $\dot X_s$, Z_s on the other hand; in order to keep things clear, we do not develop their expression further (we refer to a technical report [GM03] for full details). Finally, we approximate $H_T^{Mall.Ell.}$ by

$$\begin{aligned}H_T^{Mall.Ell.,N} = &\sum_{i=1}^d [Z_T^N\,\dot X_T^N]_i \int_0^T [\sigma^{-1}(\phi(s),X_{\phi(s)}^N)\,Y_{\phi(s)}^N]_i^*\,dW_s\\ &- \sum_{i=1}^d \int_0^T \left(\sum_j A_{\beta(j,i),T}^N B_{\beta(j,i),\phi(s)}^N\right)[\sigma^{-1}(\phi(s),X_{\phi(s)}^N)\,Y_{\phi(s)}^N]_i\,ds,\end{aligned}$$

which can be simulated using only the Brownian increments as before. We now state that the approximation above converges at order 1 w.r.t. the time step.

THEOREM 3.2. *Assume* (R_∞), (E), *and* (H). *For some* $q\geq 0$, *one has*

$$\left|\dot J(\alpha) - \mathbb{E}\left(f(X_T^N) H_T^{Mall.Ell.,N}\right)\right| \leq K(T,x)\frac{\|f\|_\infty}{T^q} h.$$

Remark 3.1. Instead of basing the computations of Malliavin derivatives for different adapted processes U, $(\mathcal{D}_{t_i} U_{t_j})_{0\le i\le j\le N}$, on the equality (2.3), an alternative approach would be to derive equations solved by $(\mathcal{D}_{t_i} U_t)_{t_i\le t\le T}$ and then approximate them with a discretization scheme (for each t_i). However, this approach would require essentially $O(N^2)$ operations, instead of $O(N)$ in our case.

3.2.2. General nondegenerate case. Denote by $0_{d_1,d_2}$ the $d_1\times d_2$ matrix with 0 for each element. Simple algebra yields that $\dot V_T^*\gamma_T^{-1}\mathcal{D}_s V_T$ is equal to

$$\begin{aligned}\dot V_T^*\gamma_T^{-1}\Pi_r^R(Y_T Z_s\sigma(s,X_s)) &= (0_{1,d-r}\dot V_T^*)\begin{pmatrix}0_{d-r,d-r} & 0_{d-r,r}\\ 0_{r,d-r} & \gamma_T^{-1}\end{pmatrix} Y_T Z_s\sigma(s,X_s)\\ &= \sum_{i=1}^d F_i[(Z_s\sigma(s,X_s))^*]_i,\end{aligned}$$

where

$$F_i = \left(Y_T^*\begin{pmatrix}0_{d-r,d-r} & 0_{d-r,r}\\ 0_{r,d-r} & \gamma_T^{-1}\end{pmatrix}\begin{pmatrix}0_{d-r,1}\\ \dot V_T\end{pmatrix}\right)_i = \sum_j U_{\kappa(i,j),T}(\gamma_T^{-1})_{\beta(i,j),\gamma(i,j)},$$

with the random variables $(U_{\kappa(i,j),T})_{i,j}$ being expressed as a product of coordinates of Y_T and $\dot V_T$. As before, we do not develop their expression to keep the formulae easy to manipulate, and we refer to [GM03] for more details.

Hence, the random variable of interest in Proposition 2.8, i.e., $H_T^{Mall.Gen.}$, is

$$\delta\left(\dot V_T^*\gamma_T^{-1}\mathcal{D}.V_T\right) = \sum_{i=1}^d F_i\int_0^T [(Z_s\sigma(s,X_s))^*]_i^* dW_s - \sum_{i=1}^d\int_0^T \mathcal{D}_s F_i[(Z_s\sigma(s,X_s))^*]_i ds.$$

By the chain rule, the Malliavin derivative of F_i is related to that of $U_{\kappa(i,j),T}$ (i.e., coordinates of Y_T and $\dot V_T$) and that of $(\gamma_T^{-1})_{\beta(i,j),\gamma(i,j)}$: the latter can be expressed in terms of γ_T^{-1} and $\mathcal{D}_s\gamma_T$ (see Lemma 2.1.6 in Nualart [Nua95, p. 89]) and we obtain

$$\text{(3.8)}\quad H_T^{Mall.Gen.} = \sum_{i,j} U_{\kappa(i,j),T}(\gamma_T^{-1})_{\beta(i,j),\gamma(i,j)}\int_0^T [(Z_s\sigma(s,X_s))^*]_i^* dW_s$$

$$\text{(3.9)}\quad -\sum_{i,j}(\gamma_T^{-1})_{\beta(i,j),\gamma(i,j)}\int_0^T \mathcal{D}_s U_{\kappa(i,j),T}[(Z_s\sigma(s,X_s))^*]_i ds$$

$$\text{(3.10)}\quad +\sum_{i,j,k,l} U_{\kappa(i,j),T}(\gamma_T^{-1})_{\beta(i,j),k}(\gamma_T^{-1})_{l,\gamma(i,j)}\int_0^T \mathcal{D}_s(\gamma_{k,l,T})[(Z_s\sigma(s,X_s))^*]_i ds.$$

Analogously to the elliptic case, the integrals above may be discretized. Furthermore, the random variables $U_{\kappa(i,j),T}$ may be approximated by $U^N_{\kappa(i,j),T}$, defined by the same product of coordinates of Y_T^N and $\dot V_T^N$ as the one defining $U_{\kappa(i,j),T}$. Its weak derivative can be computed as in (3.7): indeed, with the same arguments, one may prove that

$$\mathcal{D}_s U_{\kappa(i,j),T} = \mathbf{1}_{s\le T}\sum_k \hat U_{\kappa(i,j,k),T}\check U_{\beta(i,j,k),s}, \tag{3.11}$$

where $(\hat U_{\kappa(i,j,k),T})_{i,j,k}$ (resp., $(\check U_{\beta(i,j,k),s})_{i,j,k}$) are appropriate real values (resp., vectors) at time T (resp., at time s) of some extended systems of SDEs. Then, the natural approximation is

$$[\mathcal{D}_s U_{\kappa(i,j),T}]^N = \mathbf{1}_{s\le T} \sum_k \hat{U}^N_{\kappa(i,j,k),T} \check{U}^N_{\beta(i,j,k),s}. \tag{3.12}$$

Actually, what differs from the elliptic case are the Malliavin covariance matrix γ_T and its weak derivative. Even if $\gamma_T = \int_0^T \Pi_r^R(Y_T Z_s \sigma(s,X_s))[\Pi_r^R(Y_T Z_s\sigma(s,X_s))]^* ds$ is almost surely invertible with an inverse in any $\mathbf{L}^p$, a naive approximation may not satisfy these invertibility properties: for this reason, we add a small perturbation in its discretization as follows:

(3.13)

$$\gamma_T^N = \int_0^T \Pi_r^R(Y_T^N Z^N_{\phi(s)}\sigma(\phi(s), X^N_{\phi(s)}))[\Pi_r^R(Y_T^N Z^N_{\phi(s)}\sigma(\phi(s), X^N_{\phi(s)}))]^* ds + \frac{T}{N}\mathrm{I}_d.$$

This allows us to state the following result.

Lemma 3.3. *Assume* (R_∞) *and* (E')*. Then, for any* $p \ge 1$*, one has for some positive numbers* p_1 *and* q_1*:* $\|1/\det(\gamma_T^N)\|_{\mathbf{L}^p} \le K(T,x)\|1/\det(\gamma_T)\|^{q_1}_{\mathbf{L}^{p_1}}$ *with a constant* $K(T,x)$ *independent of* N*.*

Proof. It is easy to check that $\|\gamma_T^N - \gamma_T\|_{\mathbf{L}^p} \le K(T,x)\sqrt{h}$ (use Lemma 4.3 below). Moreover, the eigenvalues of γ_T^N are all greater than h; hence $\det(\gamma_T^N) \ge h^r$, and one deduces

$$\begin{aligned}\mathbb{E}(\det(\gamma_T^N)^{-p}) &= \mathbb{E}\left(\det(\gamma_T^N)^{-p}\mathbf{1}_{\det(\gamma_T^N)\le\frac{1}{2}\det(\gamma_T)}\right) + \mathbb{E}\left(\det(\gamma_T^N)^{-p}\mathbf{1}_{\det(\gamma_T^N)>\frac{1}{2}\det(\gamma_T)}\right)\\ &\le h^{-rp}\mathbb{P}\left(\frac{\det(\gamma_T)-\det(\gamma_T^N)}{\det(\gamma_T)} \ge \frac{1}{2}\right) + 2^p\mathbb{E}(\det(\gamma_T)^{-p})\\ &\le h^{-rp}2^q\|\,|\det(\gamma_T)-\det(\gamma_T^N)|^q\|_{\mathbf{L}^{p_1}}\|\det(\gamma_T)^{-q}\|_{\mathbf{L}^{p_2}} + 2^p\mathbb{E}(\det(\gamma_T)^{-p})\end{aligned}$$

where p_1 and p_2 are conjugate numbers. Take $q = 2rp$ to get the result. □

To deal with the weak derivative of γ_T, one needs to rewrite

$$\gamma_{k,l,T} = \sum_{i'} A_{\epsilon(k,l,i'),T}\int_0^T B_{\eta(k,l,i'),u}du,$$

where $A_{\epsilon(k,l,i'),T}$ (resp., $B_{\eta(k,l,i'),u}$) are products of coordinates of Y_T (resp., Z_u and $\sigma(u,X_u)$). As for (3.7), the Malliavin derivative of $A_{\epsilon(k,l,i'),T}$ and $B_{\eta(k,l,i'),u}$ can be expressed as

$$\mathcal{D}_s A_{\epsilon(k,l,i'),T} = \mathbf{1}_{s\le T}\sum_{j'} C_{\epsilon(k,l,i',j'),T} D_{\epsilon(k,l,i',j'),s},$$

$$\mathcal{D}_s B_{\eta(k,l,i'),u} = \mathbf{1}_{s\le u}\sum_{j'} E_{\eta(k,l,i',j'),u} F_{\eta(k,l,i',j'),s}.$$

Hence, for $s \le T$, one has

$$\begin{aligned}\mathcal{D}_s\gamma_{k,l,T} &= \sum_{i',j'} C_{\epsilon(k,l,i',j'),T}\left(\int_0^T B_{\eta(k,l,i'),u}du\right) D_{\epsilon(k,l,i',j'),s}\\ &\quad + \sum_{i',j'} A_{\epsilon(k,l,i'),T} F_{\eta(k,l,i',j'),s}\int_s^T E_{\eta(k,l,i',j'),u}du,\end{aligned} \tag{3.14}$$

which can be approximated by

$$
\begin{aligned}
[\mathcal{D}_s\gamma_{k,l,T}]^N &= \sum_{i',j'} C^N_{\epsilon(k,l,i',j'),T}\left(\int_0^T B^N_{\eta(k,l,i'),\phi(u)}du\right) D^N_{\epsilon(k,l,i',j'),s}\\
&\quad + \sum_{i',j'} A^N_{\epsilon(k,l,i'),T}\, F^N_{\eta(k,l,i',j'),s}\int_s^T E^N_{\eta(k,l,i',j'),\phi(u)}du. \qquad (3.15)
\end{aligned}
$$

We now turn to the global approximation of the weight $H_T^{Mall.Gen.}$:

(3.16)

$$
\begin{aligned}
H_T^{Mall.Gen.,N} &= \sum_{i,j} U^N_{\kappa(i,j),T}[(\gamma_T^N)^{-1}]_{\beta(i,j),\gamma(i,j)}\int_0^T [(Z^N_{\phi(s)}\sigma(\phi(s),X^N_{\phi(s)}))^*]^*_i dW_s\\
&\quad - \sum_{i,j}[(\gamma_T^N)^{-1}]_{\beta(i,j),\gamma(i,j)}\int_0^T[\mathcal{D}_{\phi(s)}U_{\kappa(i,j),T}]^N[(Z^N_{\phi(s)}\sigma(\phi(s),X^N_{\phi(s)}))^*]_i ds \qquad (3.17)\\
&\quad + \sum_{i,j,k,l} U^N_{\kappa(i,j),T}[(\gamma_T^N)^{-1}]_{\beta(i,j),k}[(\gamma_T^N)^{-1}]_{l,\gamma(i,j)}\\
&\qquad \int_0^T [\mathcal{D}_{\phi(s)}(\gamma_{k,l,T})]^N[(Z^N_{\phi(s)}\sigma(\phi(s),X^N_{\phi(s)}))^*]_i ds. \qquad (3.18)
\end{aligned}
$$

We are now in a position to state the following approximation result.

THEOREM 3.4. *Assume* (R_∞), (E'), *and* (H). *For some positive numbers p and q, one has:*

$$
\left|\dot{J}(\alpha) - \mathbb{E}(f(V_T^N)H_T^{Mall.Gen.,N})\right| \le K(T,x)\|f\|_\infty\|1/\det(\gamma_T)\|^q_{\mathbf{L}^p}h.
$$

In the hypoelliptic case (case 1) in Proposition 2.7), note that the weak approximation result above holds true under a nondegeneracy condition stated only at the initial point $(0, X_0)$, which is a significant improvement compared to [BT96a] (or more recently in [TZ04]), where the condition was stated in the whole space.

3.3. Adjoint approach. To approximate $H_T^{b,Adj.}$ and $H_T^{\sigma,Adj.}$ from Theorem 2.11, we propose the following natural estimates:

(3.19)

$$
H_T^{b,Adj.,N} = f(X_T^N)h\sum_{k=0}^{N-1}\dot{b}(t_k,X^N_{t_k})\cdot\frac{Z^{N*}_{t_k}}{T-t_k}\int_{t_k}^T[\sigma^{-1}(\phi(s),X^N_{\phi(s)})\;Y^N_{\phi(s)}]^*\;dW_s,
$$

$$
\begin{aligned}
H_T^{\sigma,Adj.,N} &= h\sum_{k=0}^{N-1}\sum_{i,j=1}^d[\dot{\sigma\sigma^*}]_{i,j}(t_k,X^N_{t_k})[f(X_T^N)-f(X^N_{t_k})]\\
&\quad\times\left(\frac{2e^j}{T-t_k}\cdot\left[Z^{N*}_{t_k}\int_{\phi(\frac{T+t_k}{2})}^T[\sigma^{-1}(\phi(s),X^N_{\phi(s)})Y^N_{\phi(s)}]^*dW_s\right]\right.\\
&\quad\times\frac{2e^i}{T-t_k}\cdot\left[Z^{N*}_{t_k}\int_{t_k}^{\phi(\frac{T+t_k}{2})}[\sigma^{-1}(\phi(s),X^N_{\phi(s)})Y^N_{\phi(s)}]^*dW_s\right]\\
&\quad\left.+\frac{2e^i}{T-t_k}\cdot\left\{\nabla_x\left[Z^{N*}_{t_k}\int_{t_k}^{\phi(\frac{T+t_k}{2})}[\sigma^{-1}(\phi(s),X^N_{\phi(s)})Y^N_{\phi(s)}]^*dW_s\right]Z^N_{t_k}e^j\right\}\right). \qquad (3.20)
\end{aligned}
$$

Derivatives $\nabla_x Y^N_{\phi(s)}$ and $\nabla_x Z^N_{t_k}$ are obtained by a direct differentiation in (3.2) and (3.3): we do not make the equations explicit; they coincide with those of the Euler procedure applied to $\nabla_x Y_t$ and $\nabla_x Z_t$ defined in (1.3) and (1.4).

These approximations also induce a discretization error in the computation of $\dot{J}(\alpha)$ of order 1 w.r.t. h.

THEOREM 3.5. *Assume* (R_∞), (E), *and* (H). *For some* $p \geq 0$, *one has*

$$\left|\dot{J}(\alpha) - \mathbb{E}\left(H_T^{b,Adj.,N} + H_T^{\sigma,Adj.,N}\right)\right| \leq K(T,x)\frac{\|f\|_\infty}{T^p}h.$$

The proof is postponed to section 4.4.

3.4. Martingale approach. The natural approximation of $H_T^{Mart.}$ defined in Theorem 2.12 may be given by

$$H_T^{Mart.,N} = \frac{f(X_T^N)}{T}\int_0^T [\sigma^{-1}(\phi(s),X^N_{\phi(s)})\dot{X}^N_{\phi(s)}]^* dW_s + \int_0^T dr\ \frac{[f(X_T^N) - f(X^N_{\phi(r)})]}{(T-\phi(r))^2}$$

$$\times \int_{\phi(r)}^T [\sigma^{-1}(\phi(s),X^N_{\phi(s)})(\dot{X}^N_{\phi(s)} - Y^N_{\phi(s)} Z^N_{\phi(r)}\dot{X}^N_{\phi(r)})]^* dW_s.$$

Unfortunately, we have not been able to analyze the approximation error $\dot{J}(\alpha) - \mathbb{E}(H_T^{Mart.,N})$ under the fairly general assumption (H$'$). Indeed, an immediate issue to handle would be to quantify the quality of the approximation of $\int_0^T dr\ \mathbb{E}\big(\frac{[f(X_T)-f(X_r)]}{(T-r)^2} \int_r^T [\sigma_s^{-1}(\dot{X}_s - Y_s Z_r \dot{X}_r)]^* dW_s\big)$ by its Riemann sum, which seems to be far from obvious under (H$'$).

4. Proof of the results on the discretization error analysis. This section is devoted to the proof of section 3's theorems analyzing the discretization error.

The trick to prove these estimates for $\mathbb{E}(f(X_T))$ usually relies on the Markov property: one decomposes the error using the PDE solved by the function $(t,x) \mapsto \mathbb{E}(f(X^x_{T-t}))$ (see Bally and Talay [BT96a]), but this makes no sense in our situation. Another way to proceed consists in cleverly using the duality relationship (2.1) with some stochastic expansion to get the right order (see Kohatsu-Higa [KH01] or [KHP02]). During the revision of this work, Kohatsu-Higa brought to our attention another paper [KHP00] where the approximation of some smooth functionals of SDEs is successfully analyzed in this way. Here, we also adopt this approach. However, the functionals of interest are much more complex. Moreover, extra technicalities compared to [KHP02] are required, because of the necessity for our Malliavin calculus computations to introduce a localization factor $\psi_T^{N,\epsilon}$.

To clarify the arguments, we first state a quite general result, whose statement enables us to reduce the proof of our theorems to check that a stochastic expansion holds true.

4.1. A more general result. By convention, we set $dW_s^0 = ds$. First, we need to define some particular forms of stochastic expansions.

DEFINITION 4.1. *The real random variable U_T (which in general depends on N) satisfies property $(\mathcal{P})$ if it can be written as*

$$U_T = \sum_{i,j=0}^{q} c_{i,j}^{U,0}(T) \int_0^T c_{i,j}^{U,1}(t) \left(\int_{\phi(t)}^t c_{i,j}^{U,2}(s) dW_s^i \right) dW_t^j$$
$$+ \sum_{i,j,k=0}^{q} c_{i,j,k}^{U,0}(T) \int_0^T c_{i,j,k}^{U,1}(t) \left[\int_0^t c_{i,j,k}^{U,2}(s) \left(\int_{\phi(s)}^s c_{i,j,k}^{U,3}(u) dW_u^i \right) dW_s^j \right] dW_t^k$$

for some adapted processes $\{(c_{i,j}^{U,i_1}(t), c_{i,j,k}^{U,i_2}(t))_{t\geq 0}\colon\ 0 \leq i,j,k \leq q, 0 \leq i_1 \leq 2, 0 \leq i_2 \leq 3\}$ (possibly depending on N) and if, for each $t \in [0,T]$, they belong to $\mathbb{D}^\infty$ with Sobolev norms satisfying $\sup_{N,t\in[0,T]} \big(\|c_{i,j}^{U,i_1}(t)\|_{k',p} + \|c_{i,j,k}^{U,i_2}(t)\|_{k',p}\big) < \infty$ for any $k', p \geq 1$.

THEOREM 4.2. *Assume* (R_∞) *and that $H_T^N - H_T$ satisfies property $(\mathcal{P})$. Then,*

1. *if f is of class C_b^3, one has*

$$\big|\mathbb{E}(f(V_T)H_T) - \mathbb{E}(f(V_T^N)H_T^N)\big| \leq K(T,x)\big(\sum_{0\leq|\alpha|\leq 3} \|\partial^\alpha f\|_\infty \big) h;$$

2. *under* (E') *and* (H)*, one has*

$$\big|\mathbb{E}(f(V_T)H_T) - \mathbb{E}(f(V_T^N)H_T^N)\big| \leq K(T,x)\ \|f\|_\infty\ \|1/\det(\gamma_T)\|_{\mathbf{L}^p}^q\ h.$$

In the statement above, $(V_t)_{0\leq t\leq T}$ corresponds to some coordinates of a SDE $(X_t)_{0\leq t\leq T}$ as it is defined in (2.5), but we can also simply consider $V = X$.

Theorem 4.2 is proved at the end of this section, and for a while, we focus on its applications to derive the announced results about the discretization errors. Remember that the approximation of the weights H is essentially based on an Euler scheme applied to a system of SDEs. For this reason, the verification of property $(\mathcal{P})$ is tightly connected to the decomposition of the error, between a Brownian SDE and its Euler approximation, in terms of a stochastic expansion. This is the purpose of the following standard lemma (for more general driven semimartingales, see Jacod and Protter [JP98]).

LEMMA 4.3. *Consider a general d'-dimensional SDE $(\bar{X}_t)_{t\geq 0}$ defined by C^∞ coefficients with bounded derivatives, and $(\bar{X}_t^N)_{t\geq 0}$ its Euler approximation:*

$$\bar{X}_t = x + \int_0^t \bar{b}(s,\bar{X}_s)\, ds + \sum_{j=1}^q \int_0^t \bar{\sigma}_j(s,\bar{X}_s)\, dW_s^j,$$
$$\bar{X}_t^N = x + \int_0^t \bar{b}(\phi(s),\bar{X}_{\phi(s)}^N)\, ds + \sum_{j=1}^q \int_0^t \bar{\sigma}_j(\phi(s),\bar{X}_{\phi(s)}^N)\, dW_s^j.$$

Then, for each t, each component of $\bar{X}_t - \bar{X}_t^N$ satisfies $(\mathcal{P})$. Namely, for $1 \leq k \leq d'$, one has

$$\bar{X}_{k,t} - \bar{X}_{k,t}^N = \sum_{i,j=0}^q c_{i,j,k}^{\bar{X},0}(t) \int_0^t c_{i,j,k}^{\bar{X},1}(s) \big(\int_{\phi(s)}^s c_{i,j,k}^{\bar{X},2}(u) dW_u^i \big) dW_s^j$$

for some adapted processes $\{(c_{i,j,k}^{\bar{X},i_1}(t))_{t\geq 0} : 0 \leq i,j \leq q, 1 \leq k \leq d', 0 \leq i_1 \leq 2\}$ satisfying $\sup_{N,t\in[0,T]} \|c_{i,j,k}^{\bar{X},i_1}(t)\|_{k',p} < \infty$ for any $k', p \geq 1$.

Proof. One has $\bar X_t-\bar X_t^N=\int_0^t \bar b'(s)(\bar X_s-\bar X_s^N)\,ds+\sum_{j=1}^q\int_0^t \bar\sigma_j'(s)(\bar X_s-\bar X_s^N)\,dW_s^j+\int_0^t[\bar b(s,\bar X_s^N)-\bar b(\phi(s),\bar X^N_{\phi(s)})]\,ds+\sum_{j=1}^q\int_0^t[\bar\sigma_j(s,\bar X_s^N)-\bar\sigma_j(\phi(s),\bar X^N_{\phi(s)})]\,dW_s^j$ with $a'(s)=\int_0^1\nabla_x a(s,\bar X_s^N+\lambda(\bar X_s-\bar X_s^N))d\lambda$ for $a=\bar b$ or $a=\bar\sigma_j$. Now, consider the unique solution of the linear equation $\mathcal{E}_t=\mathrm{I}_d+\int_0^t\bar b'(s)\mathcal{E}_s\,ds+\sum_{j=1}^q\int_0^t\bar\sigma_j'(s)\mathcal{E}_s\,dW_s^j$. From Theorem 56 p. 271 in Protter [Pro90], one deduces that

$$\begin{aligned}\bar X_t-\bar X_t^N=\mathcal{E}_t\int_0^t{\mathcal{E}_s}^{-1}\Bigg\{&[\bar b(s,\bar X_s^N)-\bar b(\phi(s),\bar X^N_{\phi(s)})]\\ &-\sum_{j=1}^q\bar\sigma_j'(s)[\bar\sigma_j(s,\bar X_s^N)-\bar\sigma_j(\phi(s),\bar X^N_{\phi(s)})]\Bigg\}\,ds\\ &+\sum_{j=1}^q\mathcal{E}_t\int_0^t{\mathcal{E}_s}^{-1}[\bar\sigma_j(s,\bar X_s^N)-\bar\sigma_j(\phi(s),\bar X^N_{\phi(s)})]\,dW_s^j;\end{aligned}$$

then, by applying Itô's formula between $\phi(s)$ and s, we can easily complete the proof of Lemma 4.3. ◻

4.2. Proof of Theorem 3.4 (general nondegenerate case). Owing to Theorem 4.2, we only have to prove that $H_T^{Mall.Gen.}-H_T^{Mall.Gen.,N}$ satisfies property $(\mathcal{P})$. Thus, it is enough to separately look at each factor in $H_T^{Mall.Gen.}$ and $H_T^{Mall.Gen.,N}$, by proving that their difference is of the form $c_{i,j}^{U,0}(T)\int_0^T c_{i,j}^{U,1}(t)\big(\int_{\phi(t)}^t c_{i,j}^{U,2}(s)dW_s^i\big)dW_t^j$ or $c_{i,j,k}^{U,0}(T)\int_0^T c_{i,j,k}^{U,1}(t)\big[\int_0^t c_{i,j,k}^{U,2}(s)\big(\int_{\phi(s)}^s c_{i,j,k}^{U,3}(u)dW_u^i\big)dW_s^j\big]dW_t^k$, while the other factors just belong to $\mathbb{D}^\infty$ with uniformly bounded Sobolev norms.

(a) The fact that the difference $U_{\kappa(i,j),T}-U^N_{\kappa(i,j),T}$ (involved in (3.8), (3.10), (3.16), and (3.18)) satisfies $(\mathcal{P})$ can be derived from an application of Lemma 4.3 by noticing that $U_{\kappa(i,j),T}$ is the product of coordinates of Y_T and $\dot V_T$.

(b) Using the expressions of γ_T and γ_T^N, one gets $\gamma_{k,l,T}-\gamma_{k,l,T}^N=-\delta_{k,l}h+\mathcal{E}_{3,1,k,l}+\mathcal{E}_{3,2,k,l}$ with

$$\begin{aligned}\mathcal{E}_{3,1,k,l}&=\int_0^T\left[\Pi_r^R(Y_TZ_s\sigma(s,X_s))[\Pi_r^R(Y_TZ_s\sigma(s,X_s))]^*\right]_{k,l}ds\\ &\quad-\int_0^T\left[\Pi_r^R(Y_T^NZ_s^N\sigma(s,X_s^N))[\Pi_r^R(Y_T^NZ_s^N\sigma(s,X_s^N))]^*\right]_{k,l}ds,\\ \mathcal{E}_{3,2,k,l}&=\int_0^T\left[\Pi_r^R(Y_T^NZ_s^N\sigma(s,X_s^N))[\Pi_r^R(Y_T^NZ_s^N\sigma(s,X_s^N))]^*\right]_{k,l}ds\\ &\quad-\int_0^T\left[\Pi_r^R(Y_T^NZ_{\phi(s)}^N\sigma(\phi(s),X_{\phi(s)}^N))[\Pi_r^R(Y_T^NZ_{\phi(s)}^N\sigma(\phi(s),X_{\phi(s)}^N))]^*\right]_{k,l}ds.\end{aligned}$$

Using Lemma 4.3 and the relation $a(s,X_s)-a(s,X_s^N)=a'(s)(X_s-X_s^N)$ with $a'(s)=\int_0^1\nabla_x a(s,X_s^N+\lambda(X_s-X_s^N))d\lambda$ available for smooth functions a, it is straightforward to see that $\mathcal{E}_{3,1,k,l}$ can be written as a sum of terms satisfying $(\mathcal{P})$. The same conclusion holds for $\mathcal{E}_{3,2,k,l}$ if we apply Itô's formula between $\phi(s)$ and s.

Finally, as $1/\det(\gamma_T)$ and $1/\det(\gamma_T^N)$ belong to any $\mathbf{L}^p$ $(p\ge 1)$ according to Lemma 3.3, it follows that the difference $[\gamma_T^{-1}]_{k',l'}-[(\gamma_T^N)^{-1}]_{k',l'}$ (involved in (3.8), (3.9), (3.10), (3.16), (3.17), and (3.18)) satisfies $(\mathcal{P})$.

(c) Concerning (3.8) and (3.16), the difference $\int_0^T [(Z_s\sigma(s,X_s))^*]_i^* dW_s - \int_0^T [(Z^N_{\phi(s)}\sigma(\phi(s),X^N_{\phi(s)}))^*]_i^* dW_s$ is equal to a sum of two terms:

$$\int_0^T [(Z_s\sigma(s,X_s))^*]_i^* dW_s - \int_0^T [(Z^N_s\sigma(s,X^N_s))^*]_i^* dW_s,$$

$$\int_0^T [(Z^N_s\sigma(s,X^N_s))^*]_i^* dW_s - \int_0^T [(Z^N_{\phi(s)}\sigma(\phi(s),X^N_{\phi(s)}))^*]_i^* dW_s.$$

It is straightforward to check that both contributions satisfy $(\mathcal{P})$, the first one because of Lemma 4.3 and the second one as an application of Itô's formula.

(d) The approximation error between terms (3.9) and (3.17) also comes from the difference $\int_0^T \mathcal{D}_s U_{\kappa(i,j),T}[(Z_s\sigma(s,X_s))^*]_i ds - \int_0^T [\mathcal{D}_{\phi(s)} U_{\kappa(i,j),T}]^N [(Z^N_{\phi(s)} \sigma(\phi(s),X^N_{\phi(s)}))^*]_i ds := \mathcal{E}_{4,1,i,j} + \mathcal{E}_{4,2,i,j}$, where

$$\begin{aligned}\mathcal{E}_{4,1,i,j} &= \int_0^T \mathcal{D}_s U_{\kappa(i,j),T}[(Z_s\sigma(s,X_s))^*]_i ds\\ &\quad - \int_0^T [\mathcal{D}_s U_{\kappa(i,j),T}]^N [(Z^N_s\sigma(s,X^N_s))^*]_i ds,\\ \mathcal{E}_{4,2,i,j} &= \int_0^T [\mathcal{D}_s U_{\kappa(i,j),T}]^N [(Z^N_s\sigma(s,X^N_s))^*]_i ds\\ &\quad - \int_0^T [\mathcal{D}_{\phi(s)} U_{\kappa(i,j),T}]^N [(Z^N_{\phi(s)}\sigma(\phi(s),X^N_{\phi(s)}))^*]_i ds.\end{aligned}$$

The error induced by the approximation between $Z_s\sigma(s,X_s)$, $Z^N_s\sigma(s,X^N_s)$, and $Z^N_{\phi(s)}\sigma(\phi(s),X^N_{\phi(s)})$ can be handled as before using Lemma 4.3 and Itô's formula. To deal with $\mathcal{D}_s U_{\kappa(i,j),T}$, $[\mathcal{D}_s U_{\kappa(i,j),T}]^N$ and $[\mathcal{D}_{\phi(s)} U_{\kappa(i,j),T}]^N$, we may recall their particular forms given by equations (3.11) and (3.12). Again, Lemma 4.3 applies to the extended systems which help in defining $\mathcal{D}_s U_{\kappa(i,j),T}$. This provides a contribution error equal to a sum of terms satisfying $(\mathcal{P})$.

(e) The difference $\int_0^T \mathcal{D}_s(\gamma_{k,l,T})[(Z_s\sigma(s,X_s))^*]_i ds - \int_0^T [\mathcal{D}_{\phi(s)}(\gamma_{k,l,T})]^N [(Z^N_{\phi(s)}\sigma(\phi(s),X^N_{\phi(s)}))^*]_i ds$ coming from (3.10) and (3.18) can be analyzed with the same arguments as before, if we take into account the specific form of the derivative $\mathcal{D}_s(\gamma_{k,l,T})$ and its approximation given by (3.14) and (3.15).

The proof of Theorem 3.4 is complete.

4.3. Proof of Theorems 3.1 (pathwise approach) and 3.2 (elliptic case).

Proof of Theorem 3.1. By an application of Theorem 4.2, it is enough to check that $\dot{X}_T - \dot{X}^N_T$ satisfies $(\mathcal{P})$, which is actually a direct consequence of Lemma 4.3.

Proof of Theorem 3.2. As for Theorem 3.4, we can check that $H_T^{Mall.Ell.} - H_T^{Mall.Ell.,N}$ satisfies $(\mathcal{P})$. Thus, Theorem 4.2 with $V_T = X_T$ and $V^N_T = X^N_T$ yields $\left|\dot{J}(\alpha) - \mathbb{E}(f(X^N_T)H_T^{Mall.Ell.,N})\right| \le \frac{K(T,x)}{T}\ \|f\|_\infty\ \|1/\det(\gamma_T)\|^q_{\mathbf{L}^p}\ h$, for some positive numbers p and q. Invoking the following well-known upper bound (see Theorem 3.5 in [KS84]) $\|1/\det(\gamma_T)\|_{\mathbf{L}^p} \le K(T,x)/T^d$ completes the estimate given in Theorem 3.2.

4.4. Theorem 3.5 (adjoint approach). The first approximation which is easy to justify is the time discretization of the integral involved in Lemma 2.9. For this, note that the function $t \mapsto \mathbb{E}(\sum_{i=1}^d \dot{b}_i(t,X_t)\partial_{x_i}u(t,X_t) + \frac{1}{2}\sum_{i,j=1}^d \dot{[\sigma\sigma^*]}_{i,j}(t,X_t)\partial^2_{x_i,x_j}u(t,X_t))$

is of class $C_b^1([0,T],\mathbb{R})$: indeed, it is a smooth function in particular because u is, and its derivatives are uniformly bounded thanks to estimates of type (2.6). Hence, it remains to prove the following upper bounds, uniformly in i,j:

(4.1)
$$\left|\mathbb{E}\left(f(X_T)\dot{b}(t_k,X_{t_k})\cdot Z_{t_k}{}^*\int_{t_k}^T[\sigma^{-1}(s,X_s)\ Y_s]^*\,dW_s - f(X_T^N)\dot{b}(t_k,X_{t_k}^N)\cdot Z_{t_k}^{N*}\right.\right.$$
$$\left.\left.\times\int_{t_k}^T[\sigma^{-1}(\phi(s),X_{\phi(s)}^N)\ Y_{\phi(s)}^N]^*\,dW_s\right)\right|\le K(T,x)\frac{\|f\|_\infty}{T^q}(T-t_k)h,$$

(4.2)
$$\left|\mathbb{E}\left([\sigma\dot{\sigma}^*]_{i,j}(t_k,X_{t_k})f(X_T)e^j\cdot\left[Z_{t_k}{}^*\int_{\frac{T+t_k}{2}}^T[\sigma^{-1}(s,X_s)Y_s]^*dW_s\right]\right.\right.$$
$$\times\ e^i\cdot\left[Z_{t_k}{}^*\int_{t_k}^{\frac{T+t_k}{2}}[\sigma^{-1}(s,X_s)Y_s]^*dW_s\right]-[\sigma\dot{\sigma}^*]_{i,j}(t_k,X_{t_k}^N)f(X_T^N)$$
$$\times\ e^j\cdot\left[Z_{t_k}^{N*}\int_{\phi(\frac{T+t_k}{2})}^T[\sigma^{-1}(\phi(s),X_{\phi(s)}^N)Y_{\phi(s)}^N]^*dW_s\right]$$
$$\left.\left.\times\ e^i\cdot\left[Z_{t_k}^{N*}\int_{t_k}^{\phi(\frac{T+t_k}{2})}[\sigma^{-1}(\phi(s),X_{\phi(s)}^N)Y_{\phi(s)}^N]^*dW_s\right]\right)\right|\le K(T,x)\frac{\|f\|_\infty}{T^q}(T-t_k)^2h,$$

(4.3)
$$\left|\mathbb{E}\left([\sigma\dot{\sigma}^*]_{i,j}(t_k,X_{t_k})f(X_T)e^i\cdot\left\{\nabla_x\left[Z_{t_k}{}^*\int_{t_k}^{\frac{T+t_k}{2}}[\sigma^{-1}(s,X_s)Y_s]^*dW_s\right]Z_{t_k}e^j\right\}\right.\right.$$
$$-\ [\sigma\dot{\sigma}^*]_{i,j}(t_k,X_{t_k}^N)f(X_T^N)e^i$$
$$\left.\left.\cdot\left\{\nabla_x\left[Z_{t_k}^{N*}\int_{t_k}^{\phi(\frac{T+t_k}{2})}[\sigma^{-1}(\phi(s),X_{\phi(s)}^N)Y_{\phi(s)}^N]^*dW_s\right]Z_{t_k}^N e^j\right\}\right)\right|$$
$$\le K(T,x)\frac{\|f\|_\infty}{T^q}(T-t_k)h.$$

Note that terms with $f(X_{t_k})$ and $f(X_{t_k}^N)$ have been removed since they do not contribute in the expectation. The three errors above can be analyzed by applying Theorem 4.2, except that the upper bounds have to include factors $(T-t_k)$ or $(T-t_k)^2$: this is a simple improvement that we won't detail here. ∎

4.5. Proof of Theorem 4.2.

4.5.1. When f is of class C_b^3. Set $V_T^{\lambda,N}=V_T^N+\lambda(V_T-V_T^N)$ for $\lambda\in[0,1]$. Then, the error to analyze is

(4.4)
$$\mathbb{E}(f(V_T)H_T)-\mathbb{E}(f(V_T^N)H_T^N)=\mathbb{E}([f(V_T)-f(V_T^N)]H_T)+\mathbb{E}(f(V_T^N)[H_T-H_T^N]).$$

Note that the difference $V_T-V_T^N$ can be expressed componentwise using Lemma 4.3; using a Taylor expansion, it follows that the first contribution in the right-hand side above can be split in a sum of terms

$$\mathcal{E}_{i,j,k} = \int_0^1 d\lambda \ \mathbb{E}\left(\partial_{x_k} f(V_T^{\lambda,N}) H_T c^0_{i,j,k}(T) \int_0^T c^1_{i,j,k}(t) \left[\left(\int_{\phi(t)}^t c^2_{i,j,k}(s) dW^i_s\right) dW^j_t\right]\right),$$

for $0 \le i, j \le q$. If i and j are different from 0, we twice apply the duality relation (2.1) combined with Fubini's theorem to obtain that $\mathcal{E}_{i,j,k}$ equals

$$\begin{aligned}
&\int_0^1 d\lambda \int_0^T dt \ \mathbb{E}\left(\mathcal{D}^j_t\big[\partial_{x_k} f(V_T^{\lambda,N}) H_T c^0_{i,j,k}(T)\big] c^1_{i,j,k}(t) \left(\int_{\phi(t)}^t c^2_{i,j,k}(s) dW^i_s\right)\right)\\
&= \int_0^1 d\lambda \int_0^T dt \int_{\phi(t)}^t ds \ \mathbb{E}\big(\mathcal{D}^i_s\big\{\mathcal{D}^j_t\big[\partial_{x_k} f(V_T^{\lambda,N}) H_T c^0_{i,j,k}(T)\big] c^1_{i,j,k}(t)\big\} c^2_{i,j,k}(s)\big)\\
&= \sum_l \int_0^1 d\lambda \ \int_0^T dt \int_{\phi(t)}^t ds \ \mathbb{E}\big(\partial_x^{\gamma_1(l)} f(V_T^{\lambda,N}) G^{1,l,\lambda,N}_{s,t,T}\big),
\end{aligned} \tag{4.5}$$

where the length of the differentiation index $\gamma_1(l)$ is less than 3. If i and/or j equals 0, an analogous formula holds with $|\gamma_1(l)| \le 2$. The random variable $G^{1,l,\lambda,N}_{s,t,T}$ is integrable with an $\mathbf{L}^1$-norm uniformly bounded w.r.t. λ, N, s, t. We have proved that this first contribution in the right-hand side of (4.4) meets the estimate of Theorem 4.2.

For the second contribution, taking into account that $H_T - H_T^N$ satisfies $(\mathcal{P})$ and using the same techniques as before, we obtain that $\mathbb{E}(f(V_T^N)[H_T - H_T^N])$ can be decomposed as a sum of terms of type

$$\int_0^T dt \int_{\phi(t)}^t ds \mathbb{E}\big(\partial_x^{\gamma_2} f(V_T^N) G^{\gamma_2,N}_{s,t,T}\big) \text{ and } \int_0^T dt \int_0^t ds \int_{\phi(s)}^s du \mathbb{E}\big(\partial_x^{\gamma_3} f(V_T^N) G^{\gamma_3,N}_{u,s,t,T}\big) \tag{4.6}$$

with $|\gamma_2| \le 2$ and $|\gamma_3| \le 3$. Uniform $\mathbf{L}^p$ estimates are available for $G^{\gamma_2,N}_{s,t,T}$ and $G^{\gamma_3,N}_{u,s,t,T}$ and the proof is complete for the case of functions f of class C^3_b.

4.5.2. When *f* is only measurable. Formally, techniques are identical, but to remove the *derivatives* of f in (4.5) and (4.6), we may integrate by parts. This step is not directly possible since the Malliavin covariance matrix of V_T^N or $V_T^{\lambda,N}$ may have bad properties, even under Assumption (E′). We do not encounter this problem in the one-dimensional elliptic case developed in [KHP02], since the convex combination of positive diffusion coefficients is still positive: this argument fails in higher dimensions with elliptic matrices, and a fortiori in a hypoelliptic framework. To circumvent this difficulty, we introduce a series of localization and approximation arguments, which unfortunately make the reading more tedious.

We put $V_T^\epsilon = V_T + \epsilon \tilde{W}_T$ and $V_T^{N,\epsilon} = V_T^N + \epsilon \tilde{W}_T$, where $(\tilde{W}_T)_{t\ge 0}$ is an extra independent r-dimensional Brownian motion and we define $V_T^{\lambda,N,\epsilon} = V_T^{N,\epsilon} + \lambda(V_T^\epsilon - V_T^{N,\epsilon})$ for $\lambda \in [0,1]$. In the following computations, the Malliavin calculus will be made w.r.t. the $(q+r)$-dimensional Brownian motion $\binom{W_t}{\tilde{W}_t}_{0\le t\le T}$.

Denote by $\bar{\mu}$ the measure defined by $\int_{\mathbb{R}^r} g(x)\bar{\mu}(dx) = \mathbb{E}(g(V_T^{0,N,0})) + \mathbb{E}(g(V_T^{1,N,0})) + \int_0^1 \mathbb{E}(g(V_T^{\lambda,N,0}))d\lambda$ and consider $(f_m)_{m\ge 1}$ a sequence of smooth functions with compact support, which converges to f in $\mathbf{L}^2(\bar{\mu})$. Thus, one easily gets that

$$\lim_{m\uparrow\infty} \lim_{\epsilon\downarrow 0} \|f_m(V_T^{\lambda,N,\epsilon})\|_{\mathbf{L}^2} = \lim_{m\uparrow\infty} \|f_m(V_T^{\lambda,N,0})\|_{\mathbf{L}^2} = \|f(V_T^{\lambda,N,0})\|_{\mathbf{L}^2} \le \|f\|_\infty \tag{4.7}$$

for $\lambda = 0$ or 1, and

$$\lim_{m\uparrow\infty}\lim_{\epsilon\downarrow 0}\left(\int_0^1 \|f_m(V_T^{\lambda,N,\epsilon})\|_{\mathbf{L}^2}d\lambda\right) = \lim_{m\uparrow\infty}\left(\int_0^1 \|f_m(V_T^{\lambda,N})\|_{\mathbf{L}^2}d\lambda\right)$$
$$\leq \lim_{m\uparrow\infty}\sqrt{\int_0^1 \mathbb{E}(f_m^2(V_T^{\lambda,N}))d\lambda} = \sqrt{\int_0^1 \mathbb{E}(f^2(V_T^{\lambda,N,0}))d\lambda} \leq \|f\|_\infty. \tag{4.8}$$

Then, the error to analyze is equal to $\mathbb{E}(f(V_T)H_T) - \mathbb{E}(f(V_T^N)H_T^N) = \lim_{m\uparrow\infty,\epsilon\downarrow 0}[\mathcal{E}_1(m,\epsilon) + \mathcal{E}_2(m,\epsilon)]$ with

$$\mathcal{E}_1(m,\epsilon) = \mathbb{E}\left(f_m(V_T^\epsilon)H_T - f_m(V_T^{N,\epsilon})H_T\right),$$
$$\mathcal{E}_2(m,\epsilon) = \mathbb{E}\left(f_m(V_T^{N,\epsilon})H_T - f_m(V_T^{N,\epsilon})H_T^N\right).$$

In view of (4.7) and (4.8), it is enough to prove the following estimates, with some constants $K(T,x)$, p and q uniform in m, and $\epsilon \leq 1$:

$$|\mathcal{E}_1(m,\epsilon)| \leq K(T,x)\Bigg(\|f_m(V_T^{0,N,\epsilon})\|_{\mathbf{L}^2} + \|f_m(V_T^{1,N,\epsilon})\|_{\mathbf{L}^2}$$
$$+ \int_0^1 \|f_m(V_T^{\lambda,N,\epsilon})\|_{\mathbf{L}^2}d\lambda\Bigg)\|1/\det(\gamma_T)\|_{\mathbf{L}^p}^q h, \tag{4.9}$$

$$|\mathcal{E}_2(m,\epsilon)| \leq K(T,x)\left(\|f_m(V_T^{0,N,\epsilon})\|_{\mathbf{L}^2} + \|f_m(V_T^{1,N,\epsilon})\|_{\mathbf{L}^2}\right)\|1/\det(\gamma_T)\|_{\mathbf{L}^p}^q h. \tag{4.10}$$

We introduce a localization factor $\psi_T^{N,\epsilon} \in [0,1]$, satisfying the following properties:

(a) $\psi_T^{N,\epsilon} \in \mathbb{D}^\infty$ and $\sup_{N,\epsilon}\|\psi_T^{N,\epsilon}\|_{k,p} \leq K(T,x)\|1/\det(\gamma_T)\|_{\mathbf{L}^{q_1}}^{q_2}$ for any integers k,p;

(b) $\mathbb{P}(\psi_T^{N,\epsilon} \neq 1) \leq K(T,x)\|1/\det(\gamma_T)\|_{\mathbf{L}^p}^q h^k$ for any $k \geq 1$, uniformly in ϵ;

(c) $\{\psi_T^{N,\epsilon} \neq 0\} \subset \{\forall\lambda \in [0,1] : \det(\gamma^{V_T^{\lambda,N,\epsilon}}) \geq \frac{1}{2}\det(\gamma^{V_T})\}$.

Its construction is given at the end of this section.

Error $\boldsymbol{\mathcal{E}_1(m,\epsilon)}$. Clearly, one has $\mathcal{E}_1(m,\epsilon) = \mathcal{E}_{1,1}(m,\epsilon) + \mathcal{E}_{1,2}(m,\epsilon)$ with

$$\mathcal{E}_{1,1}(m,\epsilon) = \mathbb{E}\left([f_m(V_T^\epsilon) - f_m(V_T^{N,\epsilon})](1-\psi_T^{N,\epsilon})H_T\right), \tag{4.11}$$

$$\mathcal{E}_{1,2}(m,\epsilon) = \mathbb{E}\left([f_m(V_T^\epsilon) - f_m(V_T^{N,\epsilon})]\psi_T^{N,\epsilon}H_T\right). \tag{4.12}$$

The first term can easily be bounded by $K(T,x)\big(\|f_m(V_T^{0,N,\epsilon})\|_{\mathbf{L}^2} + \|f_m(V_T^{1,N,\epsilon})\|_{\mathbf{L}^2}\big)$ $\|1/\det(\gamma_T)\|_{\mathbf{L}^p}^q h^k$ for any $k \geq 1$, using property (b) of $\psi_T^{N,\epsilon}$.

Now, to deal with the term $\mathcal{E}_{1,2}(m,\epsilon)$, we proceed as for the first term of the right-hand side of (4.4), that is, by decomposing $V_T^\epsilon - V_T^{N,\epsilon} = V_T - V_T^N$ using Lemma 4.3 and applying the duality relationship. Consequently, $\mathcal{E}_{1,2}(m,\epsilon)$ can be written as a sum of terms

$$\mathcal{E}_{1,2,\gamma_1}(m,\epsilon) = \int_0^1 d\lambda\int_0^T dt\int_{\phi(t)}^t ds\,\mathbb{E}\big(\partial_x^{\gamma_1}f_m(V_T^{\lambda,N,\epsilon})G_{s,t,T}^{\gamma_1,\lambda,N}\big) \tag{4.13}$$

with $|\gamma_1| \leq 3$. The random variable $G_{s,t,T}^{\gamma_1,\lambda,N}$ does not depend on ϵ and belongs to $\mathbb{D}^\infty$ with Sobolev norms uniformly bounded w.r.t. λ, N, s, t. Owing to the factor $\psi_T^{N,\epsilon}$, note that $G^{\gamma_1,\lambda,N}_{s,t,T}=0$ when $\psi^{N,\epsilon}_T=0$, because of the local property of the derivative operator (see Proposition 1.3.7 in [Nua95, p. 44]). Since $\det(\gamma^{V_T^{\lambda,N,\epsilon}})\geq\epsilon^{2r}$, one can apply Proposition 2.4, which yields

$$\mathbb{E}\big(\partial_x^{\gamma_1}f_m(V_T^{\lambda,N,\epsilon})G^{\gamma_1,\lambda,N}_{s,t,T}\big)=\mathbb{E}\big(f_m(V_T^{\lambda,N,\epsilon})H_{\gamma_1}(V_T^{\lambda,N,\epsilon},G^{\gamma_1,\lambda,N}_{s,t,T})\big)$$

for some iterated Skorohod integral $H_{\gamma_1}(V_T^{\lambda,N,\epsilon},G^{\gamma_1,\lambda,N}_{s,t,T})$. Due to the local property of the Skorohod integral (see Proposition 1.3.6 in [Nua95, p. 43]), one has $H_{\gamma_1}(V_T^{\lambda,N,\epsilon},G^{\gamma_1,\lambda,N}_{s,t,T})=H_{\gamma_1}(V_T^{\lambda,N,\epsilon},G^{\gamma_1,\lambda,N}_{s,t,T})\mathbf{1}_{\psi_T^{N,\epsilon}\neq 0}$, and applying the estimate from Proposition 2.4, one gets

$$\|H_{\gamma_1}(V_T^{\lambda,N,\epsilon},G^{\gamma_1,\lambda,N}_{s,t,T})\|_{\mathbf{L}^2}\leq C\|[\gamma^{V_T^{\lambda,N,\epsilon}}]^{-1}\mathbf{1}_{\psi_T^{N,\epsilon}\neq 0}\|^{p_3}_{\mathbf{L}^{q_3}}\|V_T^{\lambda,N,\epsilon}\|^{p_1}_{k_1,q_1}\|G^{\gamma_1,\lambda,N}_{s,t,T}\|_{k_2,q_2}$$

for some integers $p_1,p_3,q_1,q_2,q_3,k_1,k_2$. It is easy to upper bound $\|V_T^{\lambda,N,\epsilon}\|_{k_1,q_1}$ and $\|G^{\gamma_1,\lambda,N}_{s,t,T}\|_{k_2,q_2}$, uniformly in λ,N,s,t, and $\epsilon\leq 1$. It is straightforward to derive the estimation of $\|[\gamma^{V_T^{\lambda,N,\epsilon}}]^{-1}\mathbf{1}_{\psi_T^{N,\epsilon}\neq 0}\|_{\mathbf{L}^{q_3}}$ since on $\{\psi_T^{N,\epsilon}\neq 0\}$, $\det(\gamma^{V_T^{\lambda,N,\epsilon}})\geq\frac{1}{2}\det(\gamma^{V_T})$, which has an inverse in any $\mathbf{L}^p$. One has proved that

$$|\mathcal{E}_{1,2,\gamma_1}(m,\epsilon)|\leq K(T,x)\left(\int_0^1\|f_m(V_T^{\lambda,N,\epsilon})\|_{\mathbf{L}^2}d\lambda\right)\|1/\det(\gamma_T)\|^q_{\mathbf{L}^p}h;$$

this completes the estimation (4.9).

Error $\boldsymbol{\mathcal{E}_2(m,\epsilon)}$. As before, this error can be split into two parts $\mathcal{E}_2(m,\epsilon)=\mathbb{E}\big(f_m(V_T^{N,\epsilon})(1-\psi_T^{N,\epsilon})(H_T-H_T^N)\big)+\mathbb{E}\big(f_m(V_T^{N,\epsilon})\psi_T^{N,\epsilon}(H_T-H_T^N)\big)$. The first contribution can be neglected using property (b) about $\psi_T^{N,\epsilon}$. The other contribution is analyzed as the second term in the right-hand side of (4.4): it gives a sum of terms of type (4.6) with $V_T^{N,\epsilon}$ instead of V_T^N and some random variables $G^{\gamma_2,N}_{s,t,T}$ and $G^{\gamma_3,N}_{u,s,t,T}$ vanishing when $\psi_T^{N,\epsilon}=0$. Then, the rest of the proof is identical to that for (4.13); we omit details. The inequality (4.10) follows and Theorem 4.2 is proved.

Construction of $\boldsymbol{\psi_T^{N,\epsilon}}$. Set $d(\mu)=\det(\gamma^{V_T^\epsilon+\mu(V_T^N-V_T)})$ for $\mu\in[0,1]$. Since $\det(\gamma^{V_T^{\lambda,N,\epsilon}})=d(1-\lambda)$, one has

$$\det(\gamma^{V_T^{\lambda,N,\epsilon}})=\det(\gamma^{V_T^\epsilon})-\int_{1-\lambda}^1 d'(\mu)d\mu. \tag{4.14}$$

Assume that for some $C>0$, one has for any $\mu\in[0,1]$, $|d'(\mu)|^2\leq R_T^N$ with

(4.15)
$$R_T^N:=C\left(\int_0^T\|\mathcal{D}_t(V_T^N-V_T)\|^2dt\right)\left(\int_0^T\big[\|\mathcal{D}_tV_T^\epsilon\|^2+\|\mathcal{D}_t(V_T^N-V_T)\|^2\big]dt\right)^3.$$

Then, if we put $\psi_T^{N,\epsilon}=\psi(\frac{R_T^N}{\det^2(\gamma^{V_T^\epsilon})})$ with $\psi\in C_b^\infty(\mathbb{R},\mathbb{R})$ such that $\mathbf{1}_{[0,\frac{1}{8}]}\leq\psi\leq\mathbf{1}_{[0,\frac{1}{4}]}$, it is now clear that statement (a) is fulfilled using $\gamma^{V_T^\epsilon}=\gamma^{V_T}+\epsilon^2\mathrm{I}_d$. Besides, $\psi_T^{N,\epsilon}\neq 1\Rightarrow R_T^N>\frac{1}{8}\det^2(\gamma^{V_T^\epsilon})$, and thus estimates (b) follow using techniques of Lemma 3.3. Finally, $\psi_T^{N,\epsilon}\neq 0\Rightarrow R_T^N<\frac{1}{4}\det^2(\gamma^{V_T^\epsilon})\Rightarrow\det(\gamma^{V_T^{\lambda,N,\epsilon}})\geq\frac{1}{2}\det(\gamma^{V_T^\epsilon})\geq\frac{1}{2}\det(\gamma^{V_T})$ using (4.14) and (c) holds true.

It remains to prove (4.15). For this, using for any invertible matrix A the relations $\partial_A \det(A) = \det(A)[A^*]^{-1}$ (see Theorem A.98 of [RT99]) and $A^{-1} = \frac{1}{\det(A)}[\mathrm{Cof}(A)]^*$ ($\mathrm{Cof}(A)$ being the matrix of cofactors of A), one gets

$$\begin{aligned}[\det(A(\mu))]' &= \sum_{i,j} \partial_{a_{i,j}}[\det(A)]a'_{i,j}(\mu) = \sum_{i,j} \det(A)[(A^*)^{-1}]_{i,j}a'_{i,j}(\mu)\\ &= \mathrm{Tr}\big(\mathrm{Cof}(A(\mu))A^{'*}(\mu)\big).\end{aligned}$$

Put $A(\mu) = \gamma^{V_T^\epsilon + \mu(V_T^N - V_T)} = \gamma^{V_T^\epsilon} + \mu\big(\int_0^T \mathcal{D}_t V_T^\epsilon[\mathcal{D}_t(V_T^N - V_T)]^* dt + \int_0^T \mathcal{D}_t(V_T^N - V_T)[\mathcal{D}_t V_T^\epsilon]^* dt\big) + \mu^2\gamma^{V_T^N - V_T}$. We now easily prove that

$$[(\mathrm{Cof}(A))_{i,j}(\mu)]^2 \le C_1 \left(\int_0^T \big[\|\mathcal{D}_t V_T^\epsilon\|^2 + \|\mathcal{D}_t(V_T^N - V_T)\|^2\big] dt\right)^2,$$

$$[(A'_{i,j})(\mu)]^2 \le C_2 \left(\int_0^T \|\mathcal{D}_t(V_T^N - V_T)\|^2 dt\right)\left(\int_0^T \big[\|\mathcal{D}_t(V_T^N - V_T)\|^2 + \|\mathcal{D}_t V_T^\epsilon\|^2\big] dt\right).$$

Thus, (4.15) immediately follows using $d'(\mu) = [\det(A(\mu))]'$.

5. Numerical experiments.

5.1. Analysis of computational complexity.

In this paragraph we indicate the first-order approximation of the number of elementary operations (multiplications) needed for computing the different estimators w.r.t. the quantities m (number of parameters), d (dimension of the space), q (dimension of the Brownian motion), and N (number of discretization times).

In previous sections we derived estimators of the gradient of the performance measure $J(\alpha)$ w.r.t. α when J is defined by a terminal cost (see (1.2)). However, these results may be extended to functionals with instantaneous costs such as $J(\alpha) = \mathbb{E}\left(\int_0^T g(t, X_t)dt + f(X_T)\right)$ for which an estimator of the gradient may be $\frac{T}{N}\sum_{i=1}^N H_{t_i}^N(g) + H_T^N(f)$, where $H_{t_i}^N(g)$ (resp., $H_T^N(f)$) is an approximated estimator of the gradient of $\mathbb{E}(g(t_i, X_{t_i}))$ (resp., $\mathbb{E}(f(X_T))$). This case is illustrated in the first numerical experiment considered below.

The computational complexity of the different estimators depends on whether the payoff has instantaneous costs (in which case an estimator $H_{t_i}^N(g)$ for all $i \in \{1, \dots, N\}$ is needed) or if it has only a terminal cost (for which only $H_T^N(f)$ is needed). In the pathwise and Malliavin calculus methods, the cost of computing $H_{t_i}^N(g)$ for all $i \in \{1, \dots, N\}$ is the same as just computing $H_T^N(f)$, whereas in the adjoint and martingale methods, there is an additional computational burden.

- Complexity of the pathwise method: d^2qmN operations for computing the pathwise estimator $H_{t_i}^{Path.,N}(g)$ (see Proposition 1.1) for all $i \in \{1, \dots, N\}$ (required for computing $\dot{X}_{t_i}^N$, for all $i \in \{1, \dots, N\}$ and all m parameters).
- Complexity of the Malliavin calculus method, in the elliptic case ($q = d$): $3d^4(d + m)N$ operations, for computing the Malliavin calculus estimator $H_{t_i}^{Mall.Ell.,N}(g)$ (see Proposition 2.5) for all $i \in \{1, \dots, N\}$. Indeed, the complexity of computing the Malliavin derivative of each column c (among d) of Z_{c,t_i}^N is $3d^4N$ and computing the Malliavin derivative of $\dot{X}_{t_i}^N$ for all m parameters requires $3d^3mN$ operations.

- Complexity of the adjoint method: $d^4N^2 + d^2mN^2/2$ operations are needed to compute the adjoint estimator $H_{t_i}^{Adj.,N}(g) = H_{t_i}^{b,Adj.,N}(g) + \frac{1}{2}H_{t_i}^{\sigma,Adj.,N}(g)$ (see Lemma 2.9 and Theorem 2.11) for all $i \in \{1, \ldots, N\}$. Our implementation memorizes $Z_{t_i}^N$ (and other data) along the trajectory and computes $H_{t_i}^{b,Adj.,N}(g)$ and $H_{t_i}^{\sigma,Adj.,N}(g)$ for all $i \in \{1, \ldots, N\}$ afterwards. Such an implementation allows to treat problems with instantaneous costs. If we consider a problem with a terminal cost only, the complexity is reduced to $4d^4N + 3d^2mN$.
- Complexity of the martingale method: $d^2N^2/2 + dmN^2/2 + d^3mN$ for computing the martingale estimator $H_{t_i}^{Mart.,N}(g)$ (see Theorem 2.12) for all $i \in \{1, \ldots, N\}$. For problems with terminal cost only the complexity of computing $H_T^{Mart.,N}(f)$ is d^3mN.

These results are summarized in Table 5.1. Note that they are strongly related to the way we have implemented the methods and they are not guaranteed to be optimal.

Table 5.1

Complexity (in terms of number of elementary operations) of the different estimators for payoff with instantaneous costs or with terminal cost only.

	Pathwise	Malliavin	Adjoint	Martingale
Instantaneous costs	d^3mN	$3d^4(d+m)N$	$d^4N^2 + d^2m\frac{N^2}{2}$	$d(d+m)\frac{N^2}{2} + d^3mN$
Terminal cost	Same	Same	$4d^4N + 3d^2mN$	$d^2N + d^3mN$

5.2. Stochastic linear quadratic optimal control. We consider a simple one-dimensional stochastic linear quadratic (SLQ) control problems (see [CY01] and [YZ99] for an extensive study on SLQ problems) for which the control $u(\cdot)$ appears in particular in the diffusion term: $dX_t = u(t)dt + \delta u(t)dW_t$. The cost functional to be minimized is $J(u(\cdot)) = \mathbb{E}\big[\int_0^1 X_t^2 dt\big]$. This problem admits an optimal control (see references above) given by the state feedback $u^*(t) = -\frac{X_t}{\delta^2}$.

We consider a class of feedback controllers $u(t, x, \alpha)$ linearly parameterized by a three-dimensional vector α with basis functions 1, x, and t (i.e., $u(t, x, \alpha) = \alpha_1 + \alpha_2 x + \alpha_3 t$) and we write $J(\alpha)$ for $J(u(\cdot, X_\cdot, \alpha))$. In that case, the optimal control u^* belongs to the class of parameterized feedback controllers and corresponds to the parameter $\alpha^* = (0, -1/\delta^2, 0)$.

As explained before, since the payoff involves instantaneous costs, we evaluate $\nabla_\alpha J(\alpha)$ using a quantity of type $\frac{T}{N}\sum_{i=1}^N H_{t_i}^N(x^2)$. We check that the different estimators (pathwise, Malliavin calculus, adjoint, martingale) return a zero gradient for the value α^* of the parameter and we compare their variance and time for computation. Table 5.2 shows the empirical variance of the different estimators obtained for 1000 trajectories, with $h = 0.05$, $\delta = 1$. These simulations have been performed on a Pentium III, 700Mhz processor.

Table 5.2

Variance of the estimators $H^{Path.}$, $H^{Mall.Ell.}$, $H^{Adj.}$, and $H^{Mart.}$ of $\nabla_\alpha J(\alpha)$ at the optimal setting of the parameter: $\alpha_1 = 0$, $\alpha_2 = -1$, $\alpha_3 = 0$.

Var(H)	Pathwise	Malliavin	Adjoint	Martingale
α_1	0.1346	0.3754	0.6669	0.1653
α_2	0.0525	0.1188	0.1707	0.0480
α_3	0.0136	0.0446	0.0612	0.0148
CPU time	0.44s	1.95s	2.89s	0.89s

Note that the estimator used in the adjoint approach includes the term $f(X_T) - f(X_t)$ in the computation of $H_T^{\sigma,Adj.}$. Table 5.3 shows similar results for a suboptimal setting of the parameter (here α_1, α_2, and α_3 are chosen randomly within the range $[-0.1, 0.1]$). The columns Adjoint2 and Martingale2 describe simulations of the adjoint and martingale methods when the term $f(X_t)$ is omitted from the computation of the estimators $H_T^{\sigma,Adj.}$ and $H_T^{Mart.}$. We note that the variance of these estimators is significantly larger than when the term $f(X_t)$ is included, which corroborates Remark 2.1.

Table 5.3
Variance of the different estimators of $\nabla_\alpha J(\alpha)$ for $\alpha_1 = -0.0789$, $\alpha_2 = 0.0156$, $\alpha_3 = 0.0648$.

Var(H)	Pathwise	Malliavin	Adjoint	Adjoint2	Mart.	Mart.2
α_1	0.2005	4.0347	1.0287	9.5535	1.5085	4.6029
α_2	0.0252	0.6597	0.1433	1.6781	0.2360	0.7894
α_3	0.0174	0.3869	0.1051	2.2337	0.1407	1.0185
CPU time	$0.44s$	$1.97s$	$2.94s$	$2.94s$	$0.90s$	$0.90s$

For this problem with smooth cost functions, the pathwise approach provides the best performance in terms of the estimator's variance. This nice behavior for smooth costs compared to other methods has been previously observed in [FLL$^+$99].

5.2.1. Stochastic approximation algorithm. The computation of an estimator H of $\nabla_\alpha J(\alpha)$ may be used in a stochastic approximation algorithm (see, e.g., [KY97] or [BMP90]) to search a locally optimal parameterization of the controller. The algorithm begins with an initial setting of the parameter α^0. Then, if α^k denotes the value of the parameter at iteration k, the algorithm proceeds by computing an estimator $\widehat{\nabla_\alpha J(\alpha^k)}$ of $\nabla_\alpha J(\alpha^k)$ and then by performing a stochastic gradient ascent

$$\alpha^{k+1} = \alpha^k + \eta_k \widehat{\nabla_\alpha J(\alpha^k)}, \tag{5.1}$$

where the learning steps η_k satisfy a decreasing condition (for example, $\sum_k \eta_k = \infty$ and $\sum_k \eta_k^2 < \infty$; see [Pol87]). Assuming smoothness conditions on $J(\alpha)$ and a bounded variance for $\widehat{\nabla_\alpha J(\alpha^k)}$, one proves that if α^k converges, then the limit is a point of local minimum for $J(\alpha)$ (see references above for several sets of hypotheses for which the convergence is guaranteed).

Figure 5.1 illustrates this algorithm on the SLQ problem described previously, where the initial parameter is chosen randomly (same value as in Table 5.3). At iteration k, one trajectory is simulated using the controller parameterized by α^k, and an estimation $\widehat{\nabla_\alpha J(\alpha^k)}$ of $\nabla J(\alpha_k)$ (using the pathwise method) is obtained. The parameter is updated according to (5.1) with a learning step $\eta_k = \frac{K}{K+k}$. We take $K = 200$ to avoid a too rapid decreasing of $(\eta_k)_k$ at the beginning: this trick usually speeds up the numerical optimization as mentioned in [BT96b]. We note that the parameter converges to $\alpha^* = (0, -1, 0)$.

The speed of convergence for such algorithms is closely related to the gradient estimator's variance, which motivates our variance analysis for the different estimators.

5.2.2. Discretization error. Here, we report the impact of the number of discretization times in the regular mesh of the interval $[0, T]$, in the computation of the gradient $\nabla_\alpha J(\alpha)$ for the SLQ problem. Figure 5.2 reports the sensitivity of

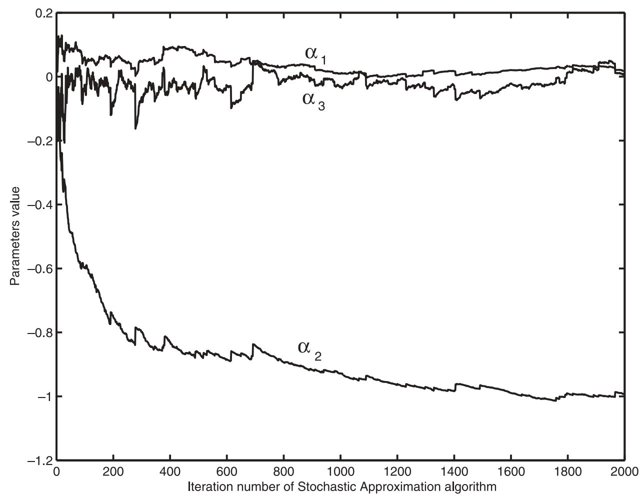

FIG. 5.1. *Stochastic approximation of the control parameters. The gradient $\nabla_\alpha J(\alpha_k)$ is estimated using the pathwise method.*

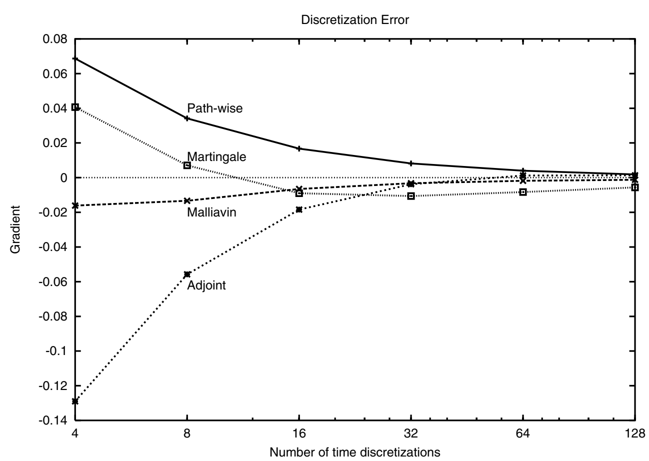

FIG. 5.2. *Discretization error as a function of the number of discretization times.*

$J(\alpha)$ (for $\alpha = \alpha^*$) w.r.t. the parameter α_1, computed with different estimators, with $N = 8, 16, 32, 64$, and 128 discretization times. Recall that, for this setting of the parameter, the true gradient is zero. To get relevant results, we have run 10^7 sample paths, which ensures that the confidence interval's width is less than 10^{-3} for all methods. We can empirically check that the convergence holds at rate $1/N$ (as previously proved), except for the martingale method, for which the rate of convergence is not clear because of the sign change (more discretization times would be needed to clarify the speed of approximation). Note that the discretization error for the Malliavin calculus estimator is smaller than for the other ones, although we have not found any explanation for this.

5.3. Sensitivity analysis in a financial market. We consider two risky assets with price process evolving according to the following SDE under the so-called riskneutral probability:

$$\frac{dS_t^1}{S_t^1} = r\,dt + \sigma(S_t^1, \lambda_1)\,dW_t^1,$$

$$\frac{dS_t^2}{S_t^2} = r\,dt + \sigma(S_t^2, \lambda_2)\left(\rho\,dW_t^1 + \sqrt{1-\rho^2}\,dW_t^2\right),$$

with constant interest rate r and volatility function $\sigma(x, \lambda) = 0.25(1 + \frac{1}{1+e^{-\lambda x}})$. The parameters of this dynamics are λ_1, λ_2, and the correlation coefficient ρ. Suppose that the true model is given by a set of parameters and that we are interested in the impact of the inaccuracy on these parameters (due to a previous statistical procedure) on option prices. For instance, we may consider digital options with payoff $\chi(S_T^1 - S_T^2)$ (where $\chi(x) = \mathbf{1}_{x\geq 0}$) whose prices are given by $J(\lambda_1, \lambda_2, \rho) = \mathbb{E}[\chi(S_T^1 - S_T^2)]$ up to the discount factor.

TABLE 5.4
Variance of the estimators $H_T^{Mall.Ell.}$, $H_T^{Adj.}$, $H_T^{Mart.}$, $H_T^{\varepsilon,Path.}$.

Var(H) or Var(H^ε)	Malliavin	Adjoint	Martingale	Pathwise $\varepsilon = 10^{-2}$	Pathwise $\varepsilon = 10^{-3}$	Pathwise $\varepsilon = 10^{-4}$
λ_1	0.0011	0.0022	0.0012	0.0053	0.0378	3.8951
λ_2	0.0048	0.0030	0.0018	0.0042	0.0296	4.9427
ρ	1.5788	2.0829	1.4323	1.6523	14.923	100.86
CPU time	20.8s	18.6s	7.31s	2.97s	2.97s	2.97s

We estimate the sensitivity of J w.r.t. the parameters λ_1, λ_2, and ρ. Table 5.4 reports the empirical variance of the estimators ($H_T^{Mall.Ell.}$, $H_T^{Adj.}$, and $H_T^{Mart.}$) of the sensitivity of J w.r.t. the parameters for the Malliavin calculus, adjoint, and martingale methods. Since the payoff function is not differentiable, we cannot directly apply the pathwise method; instead, we use χ^ε, a regularization of χ defined by $\chi^\varepsilon(x) = 1$ if $x > \varepsilon$, 0 if $x < -\varepsilon$, and $(x+\varepsilon)/(2\varepsilon)$ otherwise. Note that this induces a bias on the true value of the gradient, bias which vanishes when ε goes to 0. The pathwise estimator that we obtain with this regularization is denoted by $H^{\varepsilon,Path.}$ and Table 5.4 also reports its variance for different values of ε.

For this experiment, we ran 1000 trajectories with initial values $S_0^1 = S_0^2 = 1$, $r = 0.04$, $T = 1$, $h = 0.01$ and parameters setting $\lambda_1 = 2$, $\lambda_2 = 2$, and $\rho = 0.6$.

We note that the variance obtained by the pathwise methods is significantly larger than those obtained by the other methods (especially when ε is small), which motivates the use of the Malliavin calculus, adjoint, or martingale estimators for nonsmooth cost functions. To further reduce the variance in the case of piecewise smooth cost functions, we could combine two methods as suggested in [FLL$^+$99]: the pathwise method where the cost function is smooth and one of the other methods where it is not.

5.4. Neurocontrol for a stochastic target problem. We consider a two-dimensional stochastic target (for example, that models the displacement of a fly) moving according to a diffusion. We control a squared fly-swatter with a two-dimensional bounded force $(b(u_1), b(u_2))$ (where $u = (u_1, u_2)$ is the control), and our goal is to hit the fly at time T. Let $X = (X_1, X_2)$ be the relative coordinates of the fly w.r.t. the fly-swatter, and $V = (V_1, V_2)$ be the velocity of the fly-swatter. A simple model of the dynamics is

$$\begin{aligned} dX_{1,t} &= V_{1,t}dt + \sigma_{fly}dW_t^1, \\ dX_{2,t} &= V_{2,t}dt + \sigma_{fly}dW_t^2, \\ dV_{1,t} &= b(u_{1,t})dt + \sigma_{swat}(1 + ||u_t||)dW_t^3, \\ dV_{2,t} &= b(u_{2,t})dt + \sigma_{swat}(1 + ||u_t||)dW_t^4, \end{aligned}$$

where $b(x) = [1 - e^{-x}]/[1 + e^{-x}]$. $[(W_t^i)_{t\geq 0}]_i$ are independent standard Brownian motions; the coefficients σ_{fly} and σ_{swat} are constant. The factor $(1 + ||u||)$ (where $||u|| = \sqrt{u_1^2 + u_2^2}$) adds uncertainty on highly forced movements. The goal is to reach the fly with the fly-swatter at time T: hence, $J(u(\cdot); X_0, V_0) = \mathbb{E}[\mathbf{1}_{(X_{1,T}, X_{2,T})\in A}]$, where $A = [-a, a] \times [-a, a]$ is the squared fly-swatter.

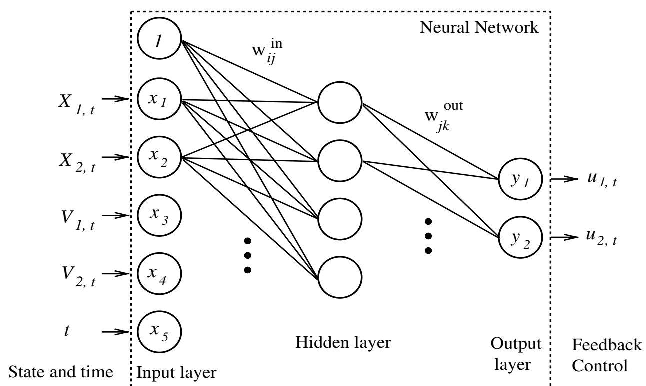

FIG. 5.3. *The architecture of the network.*

We implement the feedback controller using a one-hidden-layer neural network (see [Hay94] or [RM86] for general references on neural networks) whose architecture is given in Figure 5.3. The input layer $(x_i)_{1\leq 5}$ is connected to the state and time variables: $x_1 = X_{1,t}$, $x_2 = X_{2,t}$, $x_3 = V_{1,t}$, $x_4 = V_{2,t}$, and $x_5 = t$. There is one hidden layer with n neurons, and the output layer $(y_k)_{1\leq k\leq 2}$ returns the feedback control $y_1 = u_1(t)$, $y_2 = u_2(t)$. The network is defined by two matrices of weights (the parameters): the *input weights* $\{w_{ij}^{in}\}$ and the *output weights* $\{w_{jk}^{out}\}$. The network's output is given by $y_k = \sum_{j=1}^n w_{jk}^{out}\varphi(\sum_{i=0}^5 w_{ij}^{in}x_i)$ (for $1 \leq k \leq 2$), where the w_{0j}^{in}'s are the *bias weights* (and we set $x_0 = 1$) and $\varphi(s) = 1/(1 + e^{-s})$ is the sigmoid function.

In this experiment, we use a hidden layer with four neurons (thus there are $6 \times 4 + 4 \times 2 = 32$ control parameters); we have run 1000 trajectories with initial values of the weights chosen randomly within the range $[-0.1, 0.1]$. Here, $T = 1$, $h = 0.05$, $\sigma_{fly} = \sigma_{swat} = 0.1$, and $a = 0.1$. Each trajectory starts from a initial state chosen randomly within the range $\Omega = [-0.5, 0.5]^4$. Thus, we actually estimate $\nabla_w\mathbb{E}[J(\cdot; X_0, V_0) \mid (X_0, V_0) \sim \frac{1}{|\Omega|}\mathbf{1}_\Omega(d\omega)]$, for each weight w.

Table 5.5 reports the empirical variance of the estimators ($H^{Mall.Ell.}$, $H^{Adj.}$, and $H^{Mart.}$) of the gradient of J w.r.t. the parameters (the set of input and output weights). Here again, the function to be maximized is not differentiable and to apply the pathwise method, we use a regularization of the indicator function of A (i.e., $J^\varepsilon(\alpha) = \mathbb{E}[(\chi^\varepsilon(X_{1,T}+a)-\chi^\varepsilon(X_{1,T}-a))(\chi^\varepsilon(X_{2,T}+a)-\chi^\varepsilon(X_{2,T}-a))]$). The associated

TABLE 5.5
Variance of the estimators of the gradient of $\mathbb{E}[J(\cdot; X_0, V_0)|(X_0, V_0) \sim \frac{1}{|\Omega|}\mathbf{1}_\Omega(d\omega)]$ *w.r.t. the weights. The values provided are the averaged variances over all* 32 *parameters.*

$\mathrm{Var}(H)$ or $\mathrm{Var}(H_\varepsilon^{Path.})$	Malliavin	Adjoint	Martingale	Pathwise $\varepsilon = 10^{-3}$	Pathwise $\varepsilon = 10^{-4}$
Average over all parameters	0.1917	0.2550	0.1701	3.364	187.48
CPU time	$70.44s$	$22.04s$	$5.73s$	$2.88s$	$2.88s$

pathwise estimator is denoted $H_T^{\varepsilon,Path.}$: its variance for some values of ε is also given in Table 5.5. Although its computational time is the lowest one, the pathwise approach is not appropriate because of its large variance. On the other hand, the martingale method is the most attractive.

Stochastic approximation of an optimal controller. We run a stochastic approximation algorithm (5.1) with a learning rate $\eta_k = \frac{K}{K+k}$ (with $K = 1000$) using a neural network with four hidden neurons. At each iteration, the SA algorithm uses an estimator of the gradient of J w.r.t. the weights, which averages 50 samples of the martingale estimator.

On Figure 5.4, we plot the parameter and performance evolutions w.r.t. the iteration number: we obtain a series of weights that provide a locally optimal performance, although there is no guarantee of global optimality of the controller.

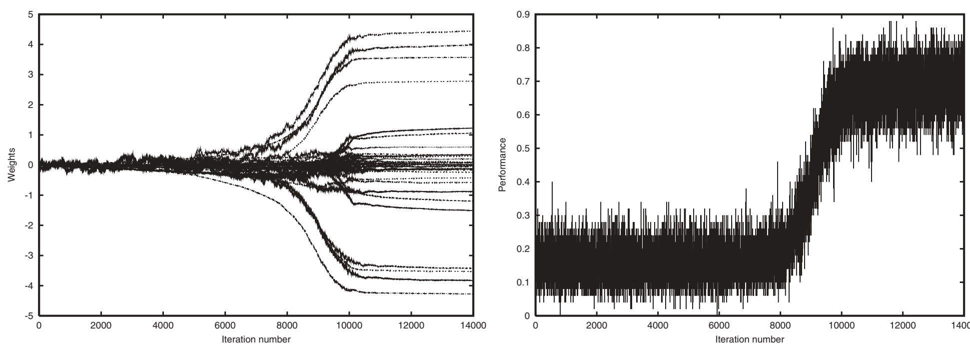

FIG. 5.4. *Stochastic approximation of the parameters (the weights of the neural network) and performance of the parameterized controller. The gradient is estimated using the martingale method.*

This stochastic gradient algorithm in the space of parameterized policies is often called *policy search* (about which an abundant literature exists in the discrete-time case; see, e.g., [BB01]), as opposed to *value search* for which some approximate dynamic programming algorithm is performed on a parameterized value function (see, e.g., [BT96b]). One may also combine these approaches and learn an approximate value function to perform a policy search (the so-called *Actor-Critic* algorithms, see e.g. [KB99]).

6. Conclusion. In this work, we have derived three new types of formulae to compute $\nabla_\alpha \mathbb{E}(f(X_T^\alpha))$ or $\nabla_\alpha \mathbb{E}(\int_0^T g(t, X_t^\alpha)dt + f(X_T^\alpha))$ using Monte Carlo methods. Our computations rely on Itô–Malliavin calculus and martingale techniques: the representations derived are simple to implement using Euler-type schemes and the associate weak error is in most of the cases linear w.r.t. the time step. We have assumed that f is bounded, but all results remain valid if f satisfies some polynomial growth.

The numerical experiments enable us to draw the following conclusions on how to select the appropriate method to use.

1. *Pathwise approach.* This can be used only if the instantaneous and terminal costs are differentiable. Otherwise, regularization procedures lead to high variances. It provides the smallest computational time. Note also that no condition on the nondegeneracy of the diffusion coefficient is needed. For the implementation, only the first derivatives of the coefficient are required.
2. *Malliavin calculus approach.* This handles the case of nonsmooth costs, but the computational time is rather large. A nondegeneracy assumption has to be satisfied but it may be not stringent (hypoellipticity, e.g.). Note that the simulation procedures require the computations of the second derivatives $\partial^2_{x,x}$ and $\partial^2_{x,\alpha}$ of the coefficients.
3. *Adjoint approach.* It can be applied in the elliptic case and is particularly efficient in terms of computational time for a large number of parameters. However, it is quite slow, especially when there are instantaneous costs (because of double time integrals and a possible large number of discretization times). The second derivatives required for the simulations concern only $\partial^2_{x,x}$.
4. *Martingale approach.* The diffusion coefficient has to be elliptic. As for the adjoint approach, it handles situations with nonsmooth costs. It appears to be very fast (almost as fast as the pathwise approach), but it is slower for instantaneous cost problems (same reason as for the adjoint approach). Note also that only the first derivatives of the coefficient are needed.

In future research, we will consider the analysis of the weak error for the martingale method and numerical optimizations in the general nondegenerate case (such as portfolio optimization problems in finance).

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