Stochastic Simultaneous Optimistic Optimization

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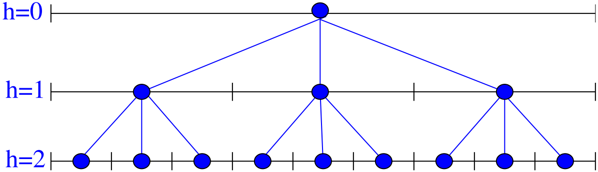
StoSOO operates on a given **hierarchical partitioning**

- For any h, $\mathcal{X}$ is partitioned in K^h cells $(X_{h,i})_{0 \leq i \leq K^h-1}$.
- K-ary tree $\mathcal{T}_\infty$ where depth $h = 0$ is the whole $\mathcal{X}$.



- StoSOO adaptively creates finer and finer partitions of $\mathcal{X}$.
- $x_{h,i} \in X_{h,i}$ is a specific state per cell where f is evaluated

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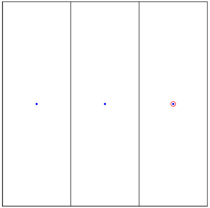

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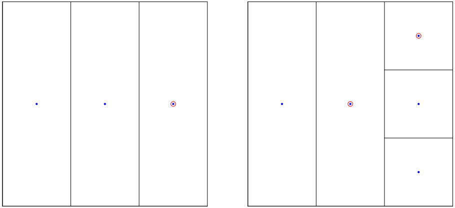

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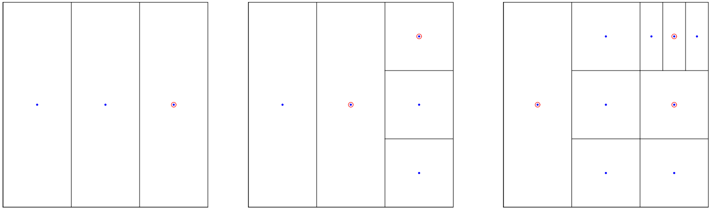

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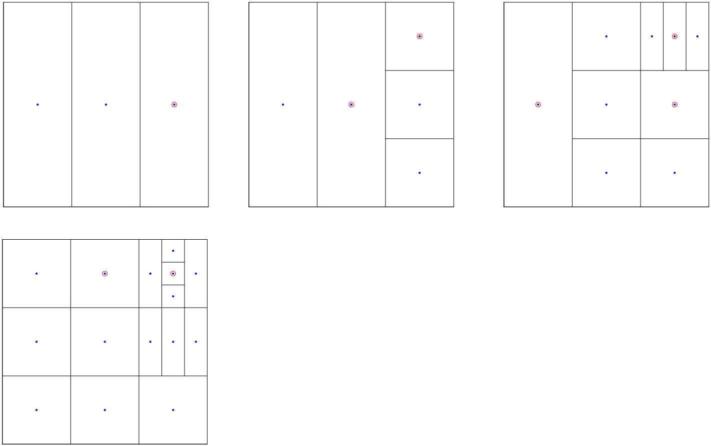

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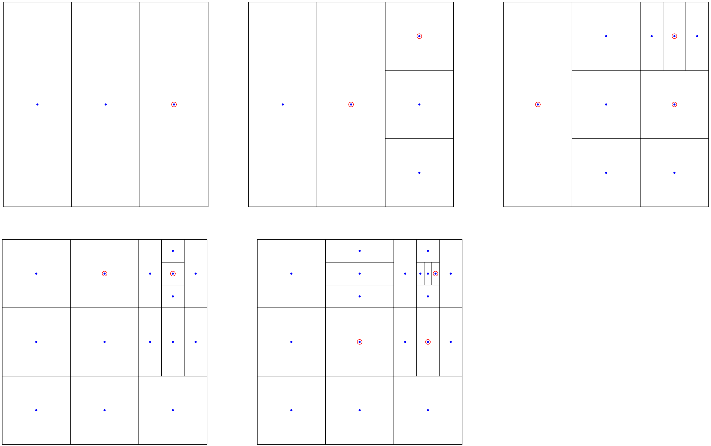

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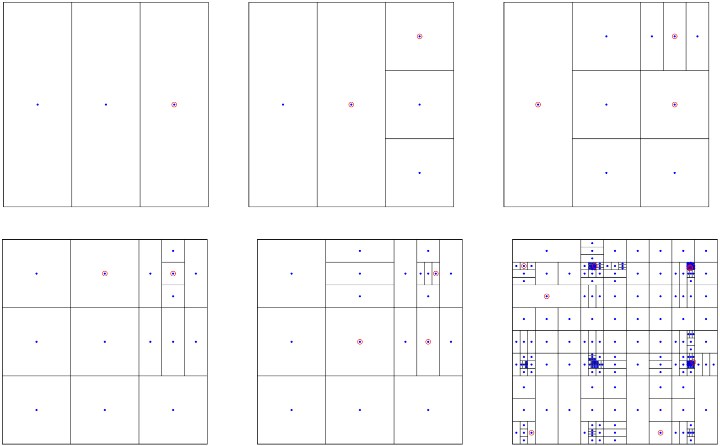
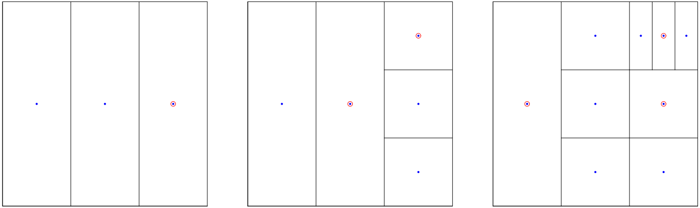


Challenge 1: **Stochasticity**

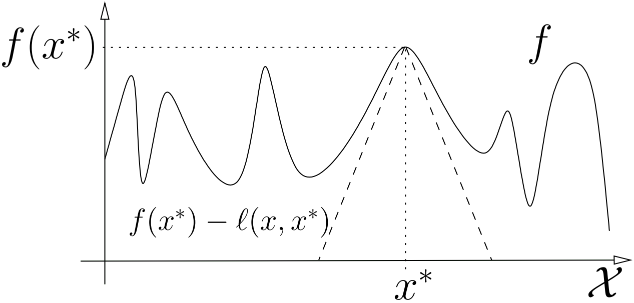


- cannot evaluate the cell only once before splitting
- cannot return the highest x_t encountered as $x(n)$

Challenge 2: **Unknown smoothness**

Assumption about the function: f **is locally smooth** w.r.t. a semi-metric ℓ around one global maximum x^*:

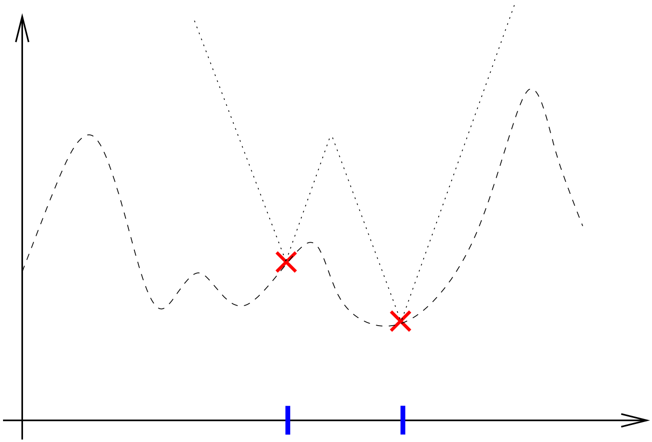
$$\forall x \in \mathcal{X} : f(x^*) - f(x) \leq \ell(x, x^*)$$



"f does not decrease too fast around x^*"

Challenge 2: **Unknown smoothness**

What can we do if the smoothness is known?



Comparison

	Deterministic function	**Stochastic** function
known smoothness	DOO	Zooming or HOO
unknown smoothness	DIRECT or SOO	

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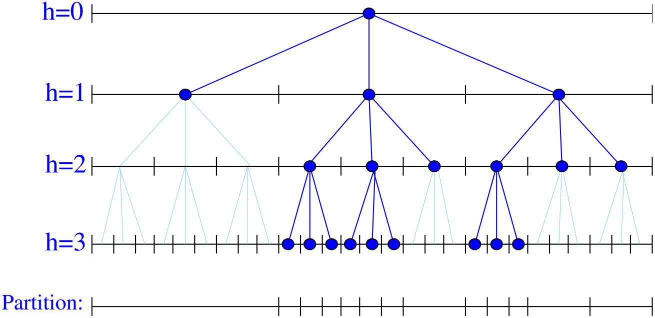

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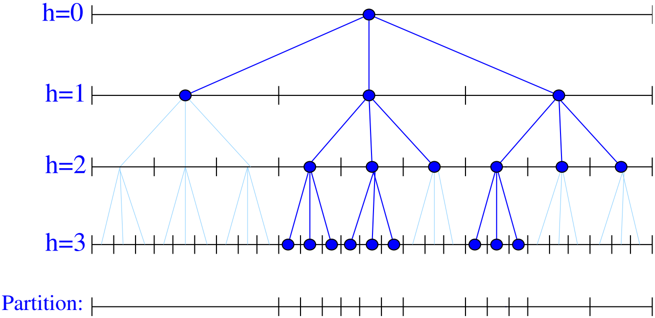

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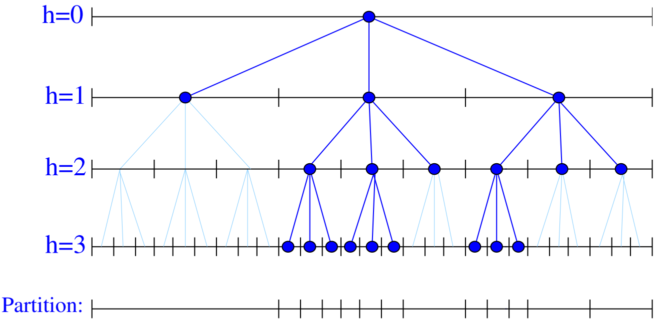

- StoSOO iteratively traverses and builds a tree over $\mathcal{X}$
- at each traversal it selects several nodes **simultaneously**
- **simultaneous** selection to consider all the leaves that can lead to potentially optimal solution

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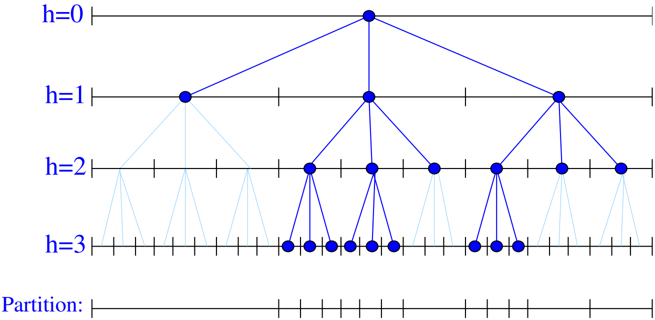

- selected nodes are either **sampled** or **expanded**

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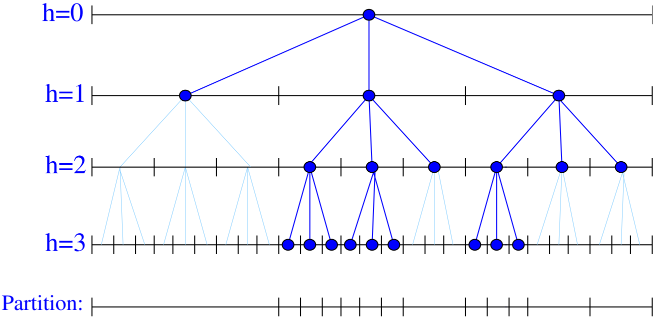

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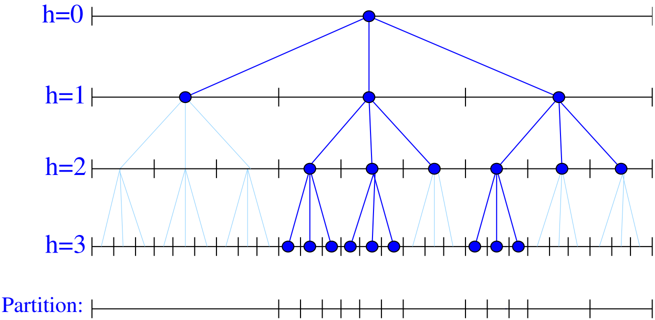

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- after sampling a leaf k times, we **expand** it

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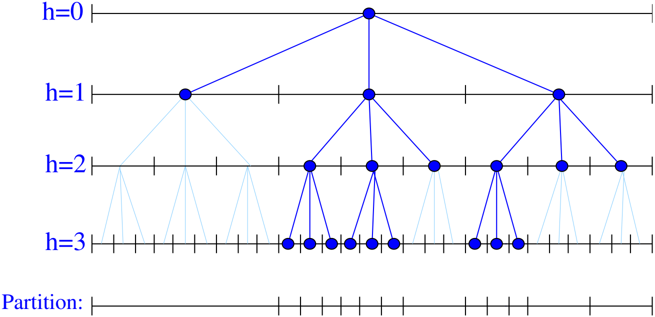

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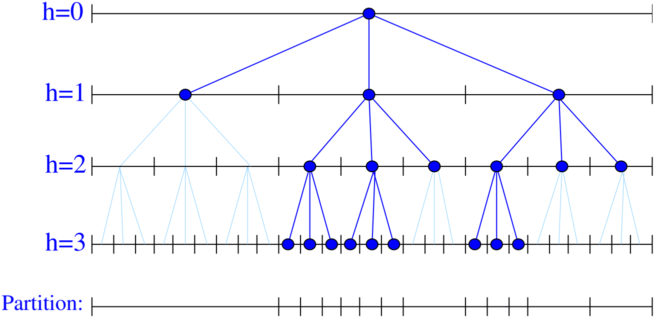

- the selection is **optimistic**, based on confidence bounds
- return the deepest **expanded** node

Dealing with stochasticity

- evaluation of f at a point x_t returns a **noisy estimate** r_t,

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$$b_{h,j}(t) \stackrel{\text{def}}{=} \hat{\mu}_{h,j}(t) + \sqrt{\frac{\log(n^2/\delta)}{2T_{h,j}(t)}}$$

where $T_{h,j}(t)$ is the number of times (h,j) has been selected up to time t, and $\hat{\mu}_{h,j}(t)$ is the empirical average of rewards

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- **optimistically** select the node with the highest b-value at each depth

Pseudocode of StoSOO

```
while t ≤ n do
  Set b_max = −∞.
  for h = 0 to maximum depth do
    Among all leaves (h, j) ∈ L_t of depth h, select
    (h, i) ∈ arg max_{(h,j)∈L_t} b_{h,j}(t)
    if b_{h,i}(t) ≥ b_max then
      Sample state x_t = x_{h,i} and collect reward r_t
      if T_{h,i}(t) ≥ k then
        Expand this node: add to T_t the K children of (h, i)
        Set b_max = b_{h,i}(t).
        Set t ← t + 1.
      end if
    end if
  end for
end while
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Return the state corresponding to the deepest expanded node:

$$x(n) = \operatorname*{arg\,max}_{x_{h,j}:(h,j)\in\mathcal{T}_n\setminus\mathcal{L}_n} h.$$

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  Set $b_{\max} = -\infty$.
  for $h = 0$ to maximum depth **do**
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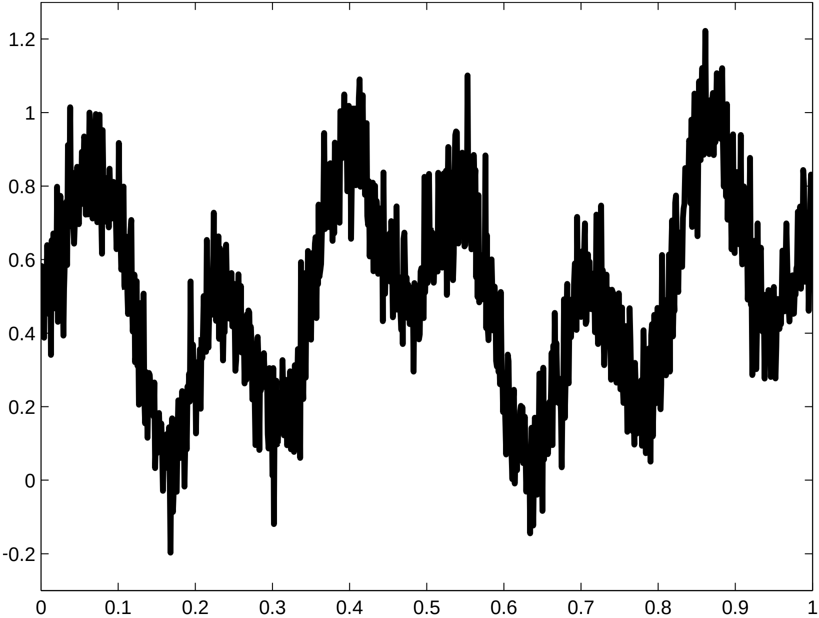
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

Measure of complexity

For any $\varepsilon > 0$, write the set of ε-optimal states:

$$\mathcal{X}_\varepsilon \stackrel{\text{def}}{=} \{x \in \mathcal{X}, f(x) \geq f^* - \epsilon\}$$

Definition (near-optimality dimension)

Smallest constant d such that there exists $C > 0$, for all $\varepsilon > 0$, the packing number of $\mathcal{X}_\varepsilon$ with ℓ-balls of radius $\nu\varepsilon$ is less than $C\varepsilon^{-d}$.

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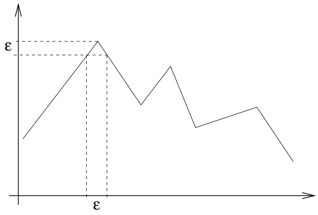
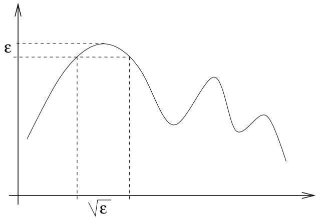
- d depends both on the function and the metric
- functions with smaller d are easier to optimize
- $d = 0$ covers a large class of functions already

Measure of complexity: Examples

$f(x^*) - f(x) = \Theta(||x^* - x||)$ $f(x^*) - f(x) = \Theta(||x^* - x||^2)$





$\ell(x, y) = ||x - y|| \rightarrow d = 0$

$\ell(x, y) = ||x - y|| \rightarrow d = D/2$
$\ell(x, y) = ||x - y||^2 \rightarrow d = 0$

Measure of complexity: Examples

StoSOO performs as if it knew the best possible semi-metric ℓ

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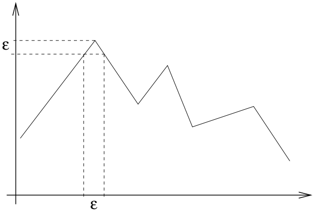

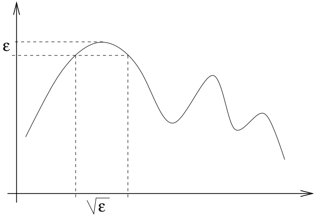

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Main result

Theorem

Let d be the $\nu/3$-near-optimality dimension and C be the corresponding constant. If the assumptions hold, then the loss of `StoSOO` *run with parameters k, $h_{\max}$, and $\delta > 0$, after n iterations is bounded, with probability $1-\delta$, as:*

$$R_n \leq 2\varepsilon + w\left(\min\left(h(n) - 1, h_\varepsilon, h_{\max}\right)\right)$$

where $\varepsilon = \sqrt{\log(nk/\delta)/(2k)}$ and $h(n)$ is the smallest $h \in \mathbb{N}$, such that:

$$C(k+1)h_{\max}\sum_{l=0}^{h}\left(w\left(l\right) + 2\varepsilon\right)^{-d} \geq n,$$

$h_\varepsilon = \arg\min\{h \in \mathbb{N} : w(h+1) < \varepsilon\}$ *and* $\sup_{x \in X_{h,i}} \ell(x_{h,i}, x) \leq w(h)$

Exponential diameters and $d = 0$

Corollary

Assume that the diameters of the cells decrease exponentially fast, i.e., $w(h) = c\gamma^h$ for some $c > 0$ and $\gamma < 1$. Assume that the $\nu/3$-near-optimality dimension is $d = 0$ and let C be the corresponding constant. Then the expected loss of StoSOO *run with parameters k, $h_{\max} = \sqrt{n/k}$, and $\delta > 0$, is bounded as:*

$$\mathbb{E}[R_n] \leq (2 + 1/\gamma)\varepsilon + c\gamma^{\sqrt{n/k}\,\min\{0.5/C, 1\} - 2} + 2\delta.$$

Exponential diameters and $d = 0$

Corollary

For the choice $k = n/\log^3(n)$ and $\delta = 1/\sqrt{n}$, we have:

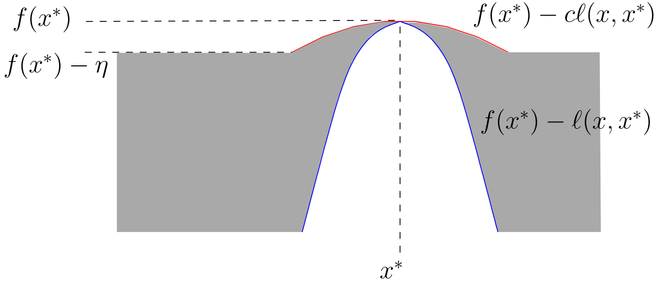
$$\mathbb{E}[R_n] = O\left(\frac{\log^2(n)}{\sqrt{n}}\right).$$

This result shows that, surprisingly, `StoSOO` can achieve the same rate $\tilde{O}(n^{-1/2})$, up to a logarithmic factor, as the HOO or Stochastic DOO algorithms run with the best possible metric, although `StoSOO` does not require the knowledge of it.

The important case $d = 0$

Let a function in such space have upper- and lower envelope around x^* of the same order, i.e., there exists constants $c \in (0, 1)$, and $\eta > 0$, such that for all $x \in \mathcal{X}$:

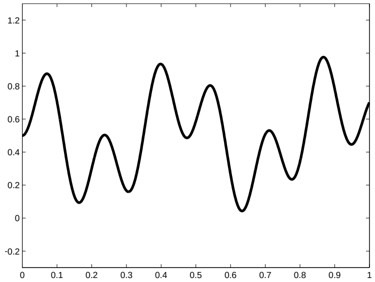
$$\min(\eta, c\ell(x, x^*)) \leq f(x^*) - f(x) \leq \ell(x, x^*). \quad (1)$$



Any function satisfying (1) lies in the gray area and possesses a lower- and upper-envelopes that are of same order around x^*.

Two-sine product function

$$f_1(x) = \frac{1}{2}\sin(13x)\cdot\sin(27x)$$



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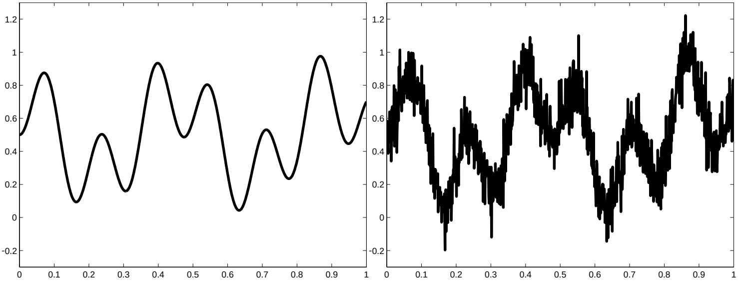

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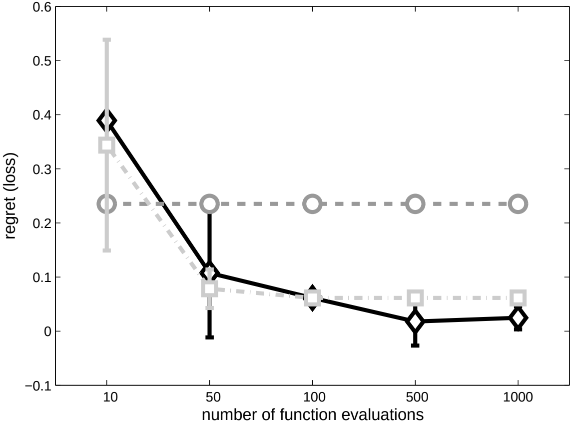
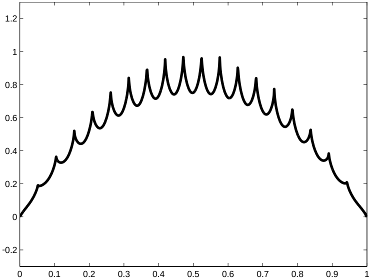


`StoSOO` (diamonds) vs.
Stochastic DOO with ℓ_1 (circles) and ℓ_2 (squares) on f_1

Garland function

$$f_2(x) = 4x(1-x)\cdot\left(\frac{3}{4}+\frac{1}{4}\left(1-\sqrt{|\sin(60x)|}\right)\right).$$



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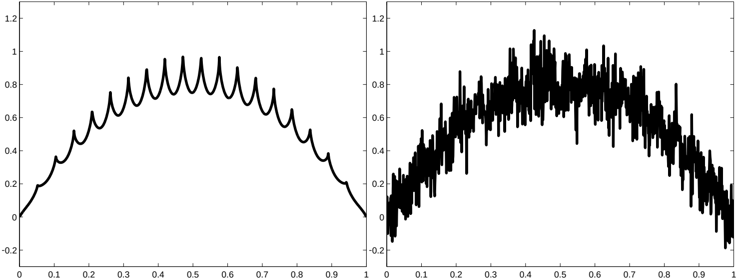

Not Lipschitz for any L!

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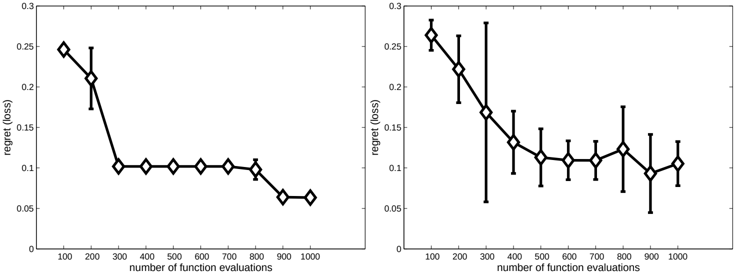

StoSOO's performance for the garland function.
Left noised with $\mathcal{N}_T(0, 0.01)$. **Right**: Noised with $\mathcal{N}_T(0, 0.1)$.

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Thank you!

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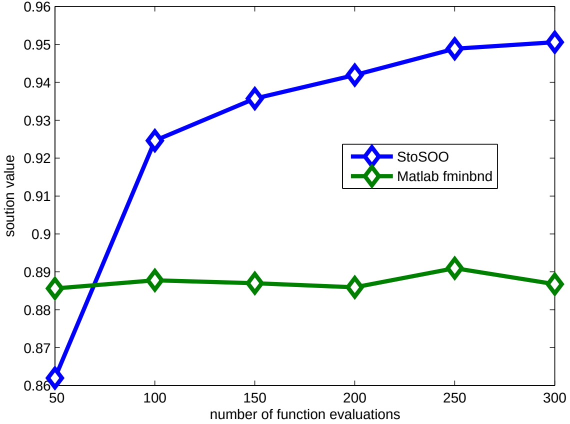
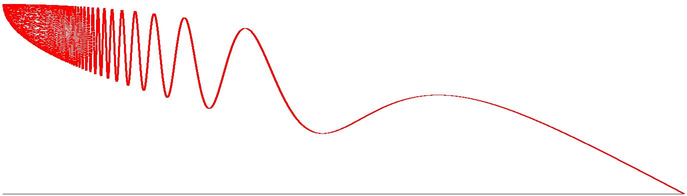
Noised two-sine product function: StoSOO vs. MATLAB



When $d > 0$?

Example of a function with different order in the upper and lower envelopes, when $\ell(x, y) = |x - y|^\alpha$:

$$f(x) = 1 - \sqrt{x} + (-x^2 + \sqrt{x}) \cdot (\sin(1/x^2) + 1)/2$$



The lower-envelope behaves like a square root whereas the upper one is quadratic. There is no semi-metric of the form $|x - y|^\alpha$ for which $d < 3/2$.