Fast sampling of projection determinantal point processes

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Motivation

Image search

Relevance only:

Relevance + diversity:

Figure 1: Jaguar query, from A. Kulesza's slides

Motivation

Text summarization (Kulesza and Taskar, 2012)

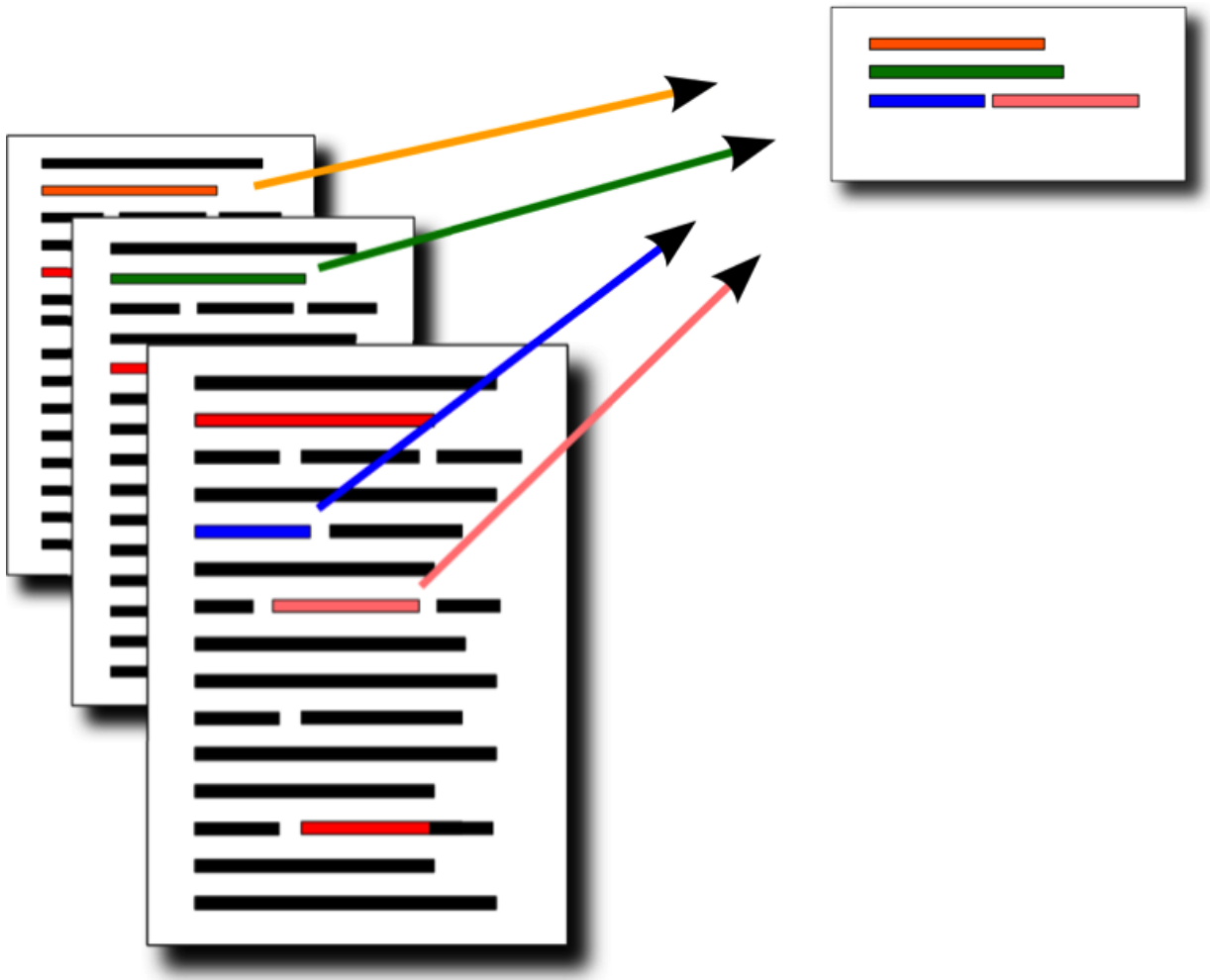


Figure 1: Extractive summarization

Definition

- $\{1, \dots, N\}$ indices/labels of items
- $\mathbf{K} \succeq \mathbf{0}_N$ similarity matrix
- DPP($\mathbf{K}$) measure on subsets of $\{1, \dots, N\}$
- $\mathcal{X} \sim \text{DPP}(\mathbf{K})$ if $\forall S \subseteq \{1, \dots, N\}$,

$$\mathbb{P}[S \subseteq \mathcal{X}] = \det \mathbf{K}_S$$

- Existence is guaranteed when $\mathbf{0}_N \preceq \mathbf{K} \preceq \mathbf{I}_N$

Diversity selection

- If $\mathcal{X} \sim \text{DPP}(\mathbf{K})$, then $\forall i, j$

$$\begin{aligned}\mathbb{P}\left[\{i,j\} \subseteq \mathcal{X}\right] &= \begin{vmatrix} \mathbb{P}\left[i \in \mathcal{X}\right] & \mathbf{K}_{ij} \\ \mathbf{K}_{ij} & \mathbb{P}\left[j \in \mathcal{X}\right] \end{vmatrix} \\ &= \mathbb{P}\left[i \in \mathcal{X}\right]\mathbb{P}\left[j \in \mathcal{X}\right] - \mathbf{K}_{ij}^2 \\ &\leq \mathbb{P}\left[i \in \mathcal{X}\right]\mathbb{P}\left[j \in \mathcal{X}\right]\end{aligned}$$

- $|\mathbf{K}_{ij}| \approx$ similarity between i and j
  - The larger $|\mathbf{K}_{ij}|$ the smaller $\mathbb{P}\left[\{i,j\} \subseteq \mathcal{X}\right]$
  - Diversity/repulsion
  - $|\mathbf{K}_{ij}|$ yields departure from independence

DPPs intrinsically capture diversity/repulsion!

Some examples

Carries

X_i	0	3	6	5	4	7	3	0	9
$X_{i-1} + X_i \mod b$		3	9	4	8	5	8	8	7
C_i		○	○	●	○	●	○	○	●

$$\mathbb{P}[C_i = \bullet] = \frac{1}{2}\left(1 - \frac{1}{b}\right)$$

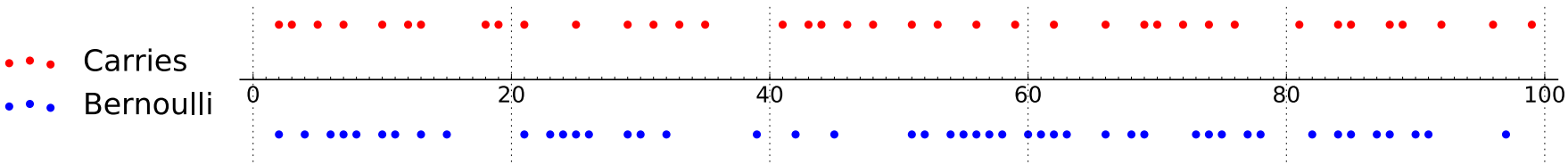


Figure 2: Carries process

Uniform spanning trees (Lyons, 2003)

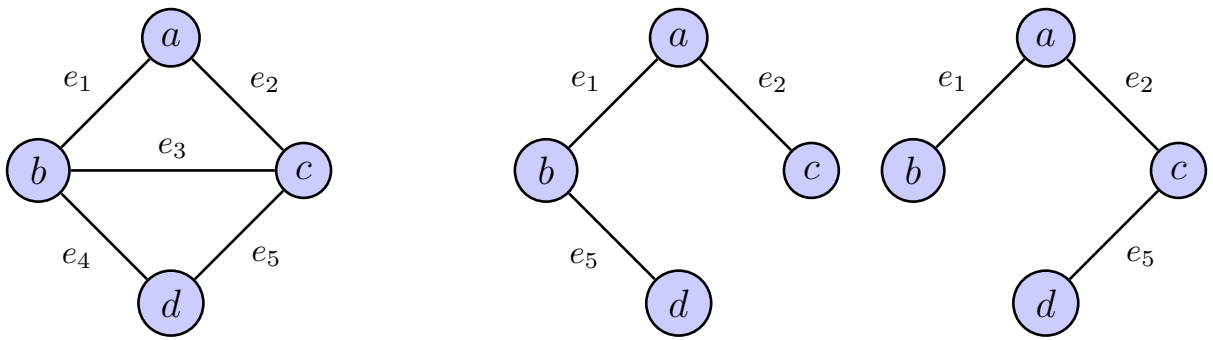


Figure 3: A graph and 2 of its spanning trees

Projection DPPs

$$\mathbf{K} = \sum_{i=1}^{r} u^{(i)} u^{(i)\mathsf{T}}$$

- $\operatorname{Sp} \mathbf{K} = \{0, 1\}$
- $|\mathcal{X}| \overset{a.s.}{=} \operatorname{Tr} \mathbf{K} = \operatorname{rk} \mathbf{K} = r$
- Projection DPPs sample r-subsets of items:
  - Summaries made of r sentences
  - Bags of r images

Setup

In the following, we build the $r \times N$ feature matrix

$$\mathbf{A} = \left(\sqrt{q_1}\phi_1 | \dots | \sqrt{q_N}\phi_N\right)$$

And construct the projection kernel onto $\operatorname{Im} \mathbf{A}^\top$

$$\mathbf{K} = \mathbf{A}^\top \left[\mathbf{A}\mathbf{A}^\top\right]^{-1} \mathbf{A}$$

- $\operatorname{rk} \mathbf{K} = r$

Let $B = \{i_1, \dots, i_r\}$, then

$$\mathbb{P}\left[B \subseteq \mathcal{X}\right] = \mathbb{P}\left[\mathcal{X} = B\right] = \frac{\left|\det \mathbf{A}_{:B}\right|^2}{\det \mathbf{A}\mathbf{A}^\top}$$

Notation

- $\mathbf{A}_{:B} \triangleq \mathbf{B}$
- $\mathbb{P}\left[\mathcal{X} = B\right] \propto \operatorname{Vol}^2 \left\{\sqrt{q_i}\phi_i \,;\, i \in B\right\} \triangleq \operatorname{Vol}^2 \mathbf{B}$
- $\mathcal{B} \triangleq \{B \,;\, |B| = r, \det \mathbf{B} \neq 0\}$, collection of cols of $\mathbf{A}$ forming a basis of $\mathbb{R}^r$
- DPP($\mathbf{K}$) has support $\mathcal{B}$

Exact sampling procedure

- $\mathbf{K}$ is a (orthogonal) projection matrix

$$\begin{aligned}\mathbf{K} &= \mathbf{A}^\mathsf{T}\left[\mathbf{A}\mathbf{A}^\mathsf{T}\right]^{-1}\mathbf{A}\\ &= \sum_{i=1}^{r} u^{(i)}\,u^{(i)\mathsf{T}}\\ &= \mathbf{\Phi}^\mathsf{T}\mathbf{\Phi}\end{aligned}$$

- $\mathbf{\Phi}$ is a $r \times N$ feature matrix with

$$\varphi_n = \left(u_n^{(1)}, \ldots, u_n^{(r)}\right)^\mathsf{T}$$

- How to sample B with

$$\begin{aligned}\mathbb{P}\left[\mathcal{X} = B\right] &= \det \mathbf{K}_B\\ &= \left|\det \mathbf{\Phi}_{:\mathbf{B}}\right|^2 = \mathrm{Vol}^2\left\{\varphi_i\,;\, i \in B\right\}\\ &\propto \left|\det \mathbf{A}_{:B}\right|^2 = \mathrm{Vol}^2\left\{\sqrt{q_i}\phi_i\,;\, i \in B\right\}\end{aligned}$$

Exact sampling procedure

- Chain rule $\approx$ Gram-Schmidt orthogonalization of $\mathbf{\Phi}$

$$\begin{aligned}
\mathbb{P}\left[i_1 = i\right] &\propto \left\|\varphi_i\right\|^2 \\
\mathbb{P}\left[i_2 = i | i_1\right] &\propto \mathsf{dist}^2\left(\varphi_i, \mathsf{Span}\left\{\varphi_{i_1}\right\}\right) \\
&\vdots \\
\mathbb{P}\left[i_r = i | i_{1:r\text{-}1}\right] &\propto \mathsf{dist}^2\left(\varphi_i, \mathsf{Span}\left\{\varphi_{i_1}, \ldots, \varphi_{i_{r-1}}\right\}\right)
\end{aligned}$$

- equivalent to *Base* $\times$ *Height* formula

$$\begin{aligned}
\mathbb{P}\left[i_1, \ldots, i_r\right] &\propto \mathsf{Vol}^2\left\{\varphi_{i_1}, \ldots, \varphi_{i_r}\right\} = \left|\det \mathbf{\Phi}_{:i_{1:r}}\right|^2 \\
&\propto \mathsf{Vol}^2\left\{\sqrt{q_{i_1}}\phi_{i_1}, \ldots, \sqrt{q_{i_r}}\phi_{i_r}\right\} = \left|\det \mathbf{A}_{:i_{1:r}}\right|^2
\end{aligned}$$

Exact sampling procedure

- Chain rule $\approx$ Gram-Schmidt orthogonalization of $\mathbf{\Phi}$

$$\mathbb{P}\left[i_1 = i\right] \propto \mathbf{K}_{ii}$$

$$\mathbb{P}\left[i_2 = i \middle| i_1\right] \propto \mathbf{K}_{ii} - \frac{\left|\mathbf{K}_{i_1 i}\right|^2}{\mathbf{K}_{i_1 i_1}}$$

$$\vdots$$

$$\mathbb{P}\left[i_r = i \middle| i_{1:r\text{-}1}\right] \propto \mathbf{K}_{ii} - \mathbf{K}_{i_{1:r-1} i}{}^{\mathsf{T}} \left[\mathbf{K}_{i_k i_l}\right]_{k,l=1}^{r\text{-}1\,-1} \mathbf{K}_{i_{1:r-1} i}$$

- equivalent to *Base* $\times$ *Height* formula

$$\begin{aligned}\mathbb{P}\left[i_1, \ldots, i_r\right] &\propto \mathsf{Vol}^2\left\{\varphi_{i_1}, \ldots, \varphi_{i_r}\right\} = \left|\det \mathbf{\Phi}_{:i_{1:r}}\right|^2 \\ &\propto \mathsf{Vol}^2\left\{\sqrt{q_{i_1}}\phi_{i_1}, \ldots, \sqrt{q_{i_r}}\phi_{i_r}\right\} = \left|\det \mathbf{A}_{:i_{1:r}}\right|^2\end{aligned}$$

Sampling scheme: interpretation

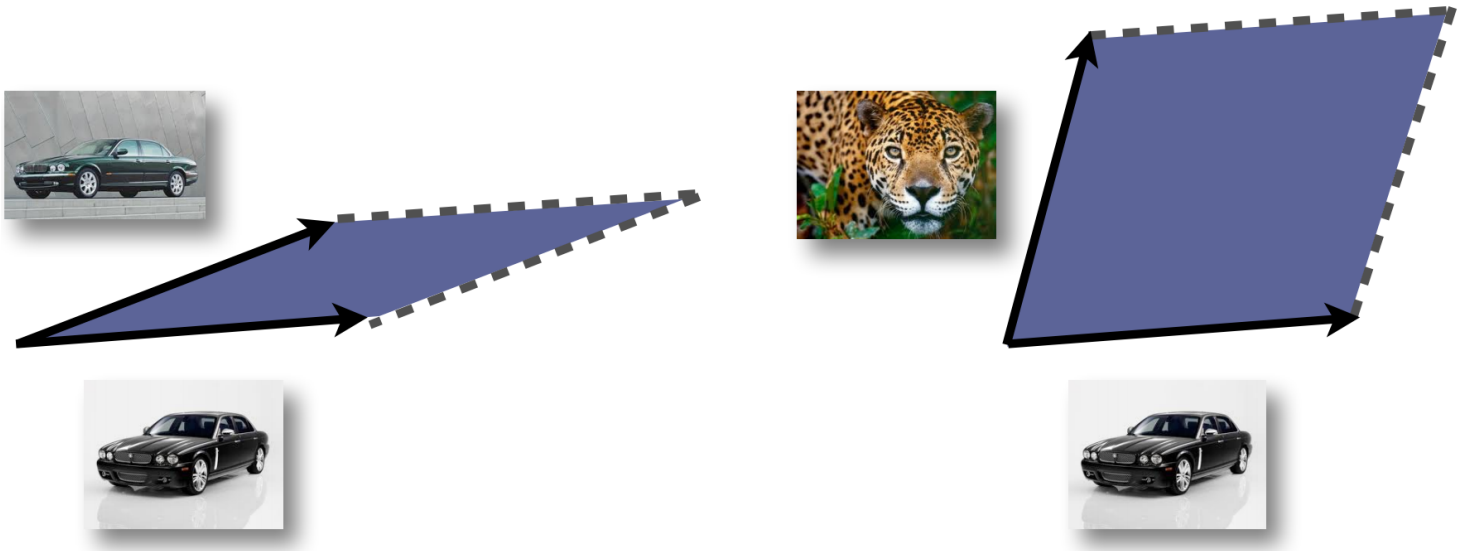
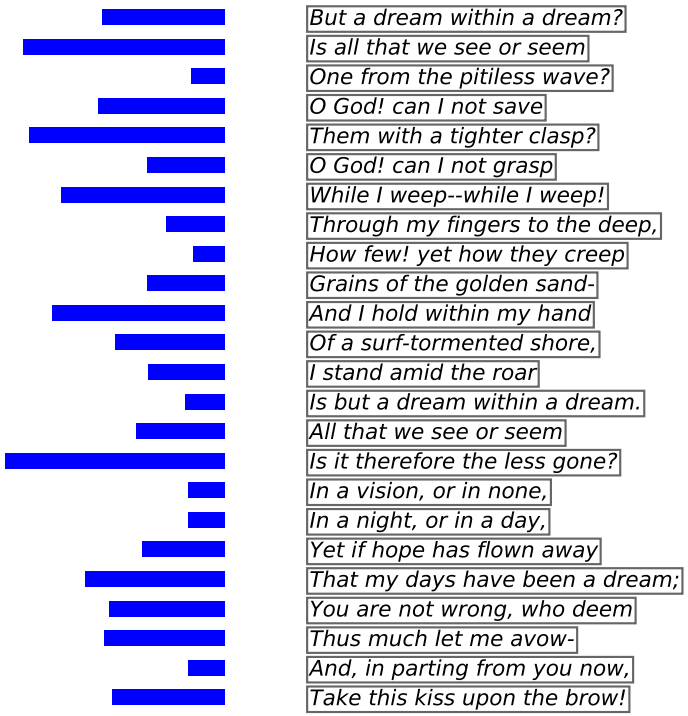


Figure 4: Diversity via squared volumes, from A. Kulesza's slides

Cost ☹: Eigen-decomposition $= \mathcal{O}(N^3)$ and Gram-Schmidt $= \mathcal{O}(rN^2)$ §

Exact sampling: 2D visualisation



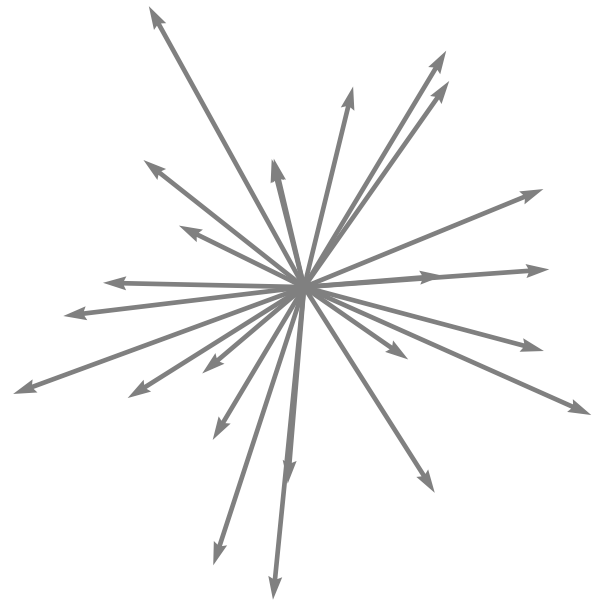
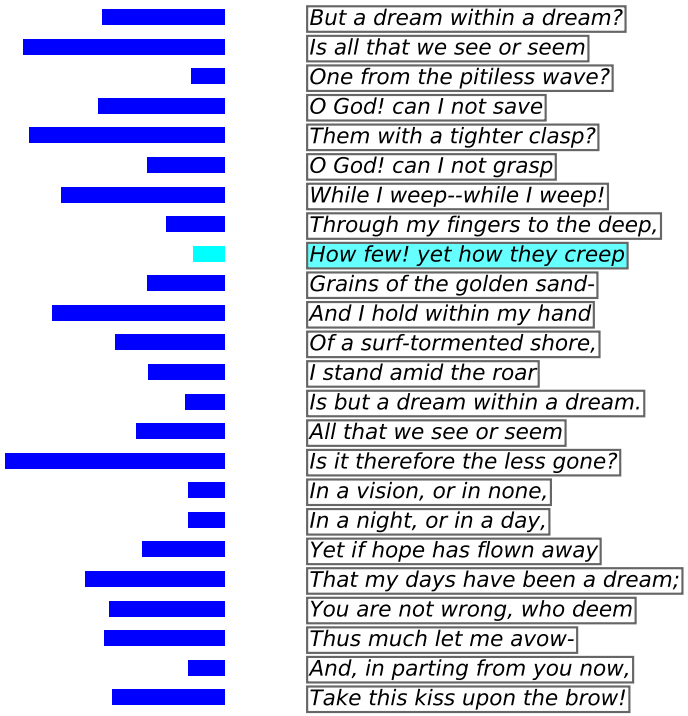


Figure 5

$$\begin{pmatrix} 0.68 & 0.39 & 0.67 & 0.4 & 0.4 & 0.14 & -0.09 & -0.08 & -0.43 & -0.45 & -0.35 & -0.56 & -0.67 & -0.81 & -0.49 & \cdots \\ 0.05 & 0.03 & 0.3 & 0.63 & 0.72 & 0.61 & 0.39 & 0.39 & 0.86 & 0.39 & 0.19 & 0.01 & -0.09 & -0.33 & -0.34 & \cdots \end{pmatrix}$$

Exact sampling: 2D visualisation



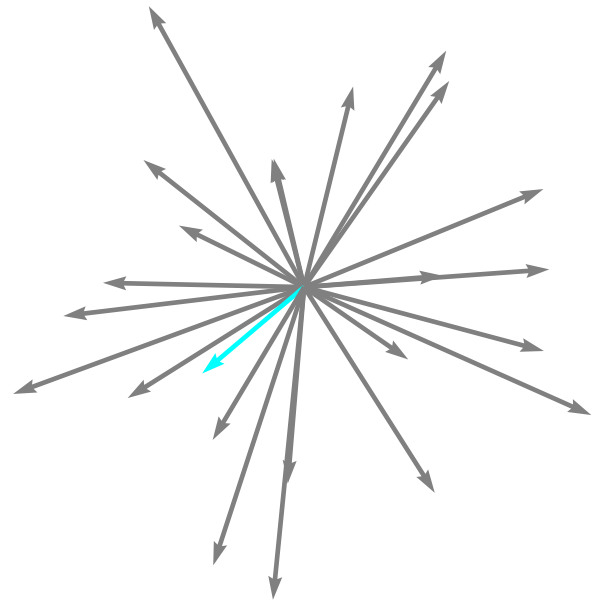
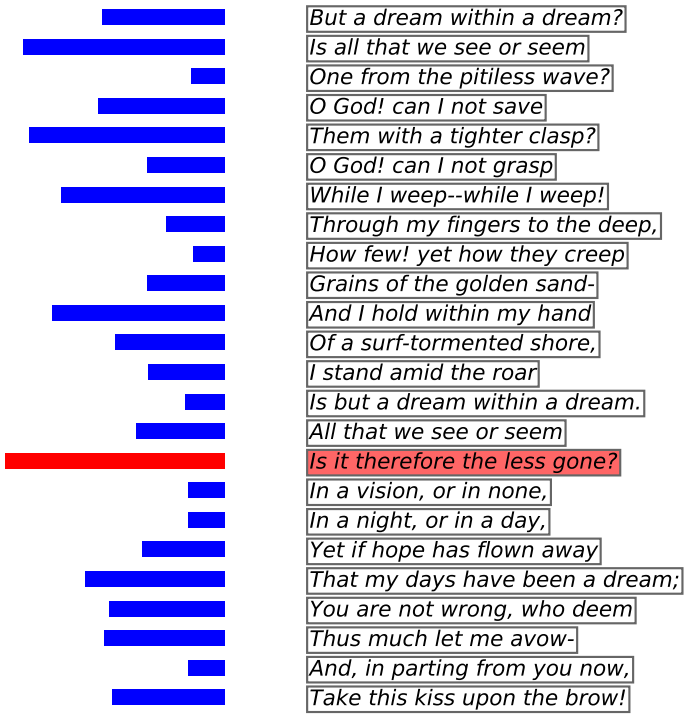


Figure 5: φ_i with smallest $\|\cdot\|^2$

$$\mathbb{P}\left[i_1 = i\right] \propto \|\varphi_i\|^2$$

Exact sampling: 2D visualisation



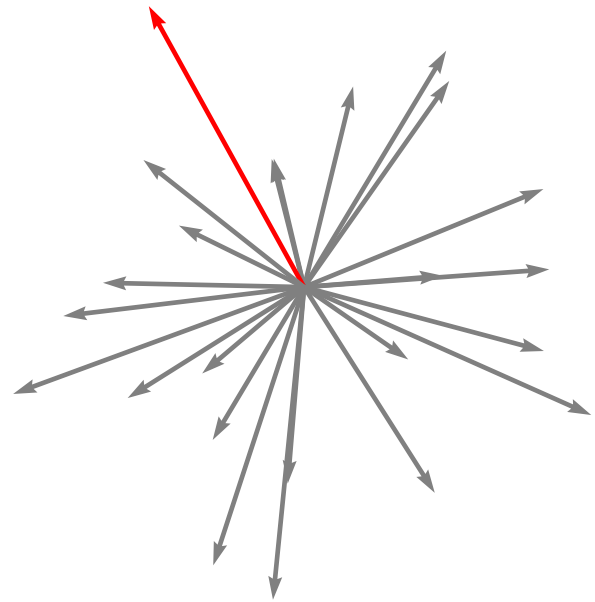
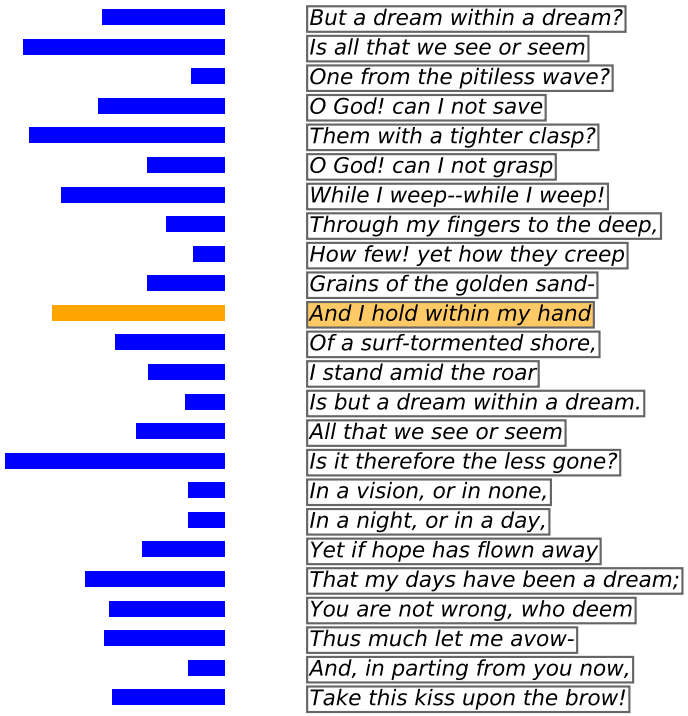


Figure 5: φ_i with largest $\|\cdot\|^2$

$$\mathbb{P}\left[i_1 = i\right] \propto \|\varphi_i\|^2$$

Exact sampling: 2D visualisation



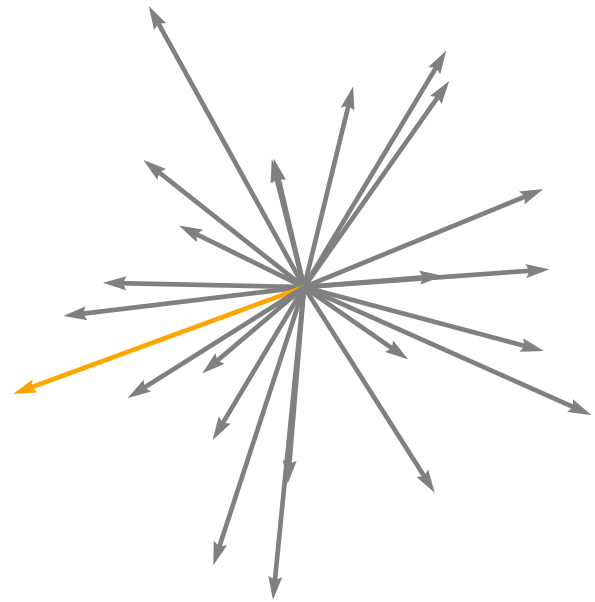
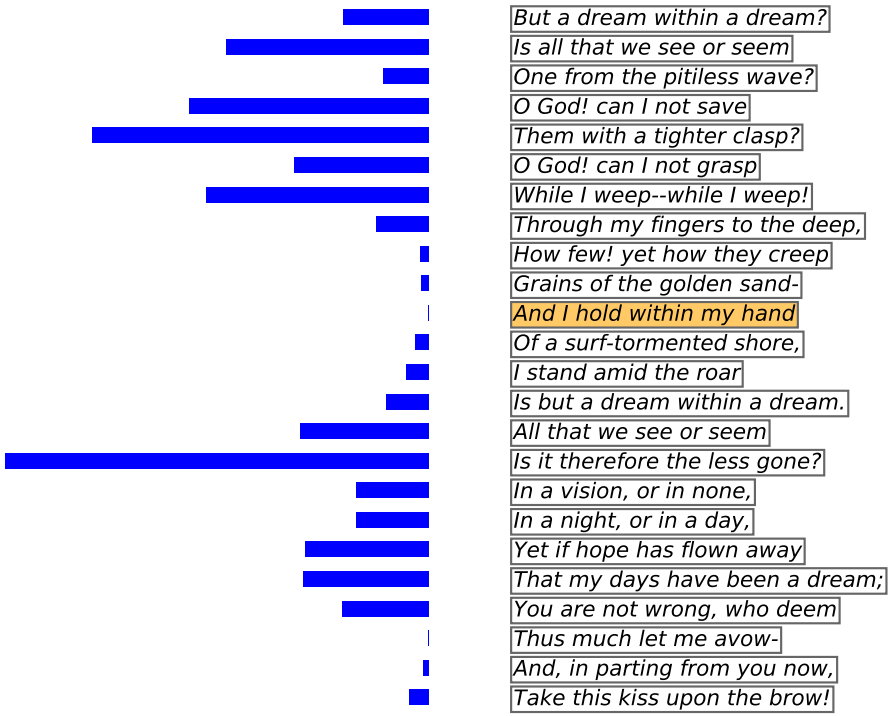


Figure 5: Feature vector selected

$$\mathbb{P}\left[i_1 = i\right] \propto \left\|\varphi_i\right\|^2$$

Exact sampling: 2D visualisation



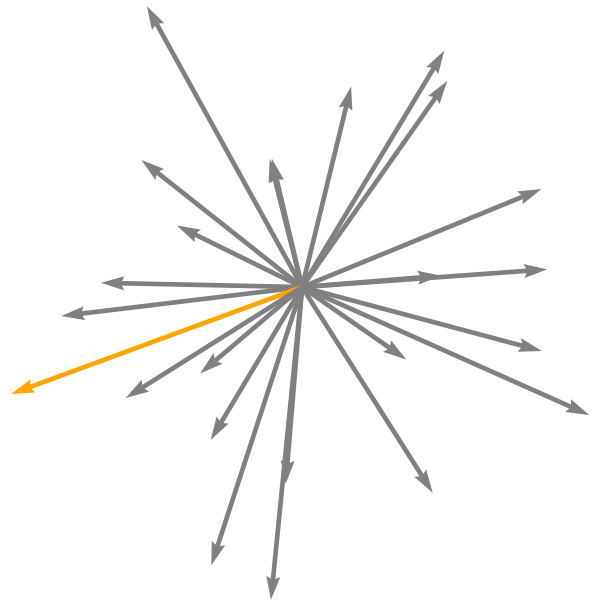
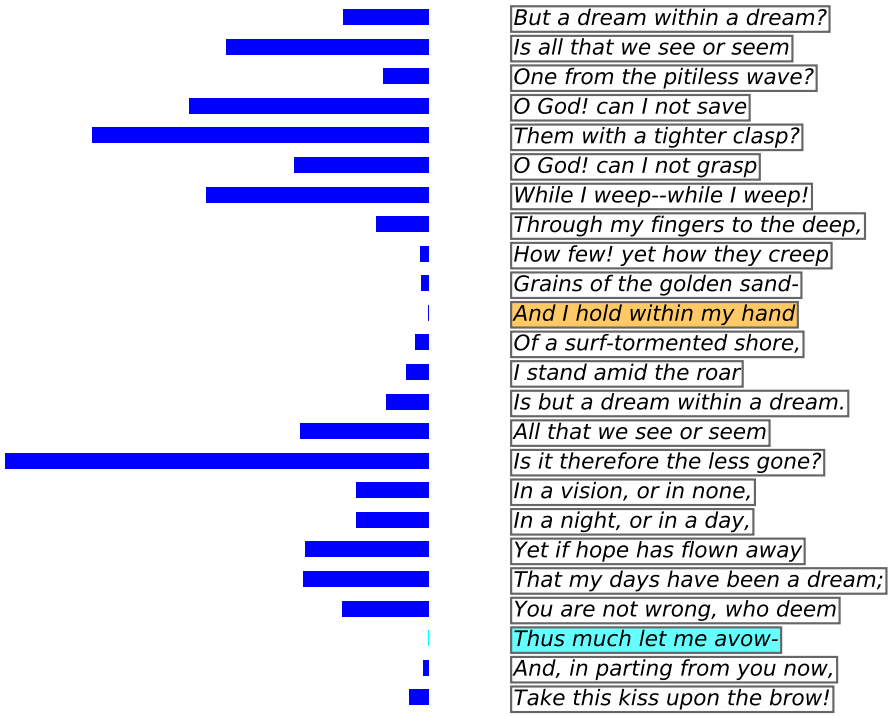


Figure 5: Update the conditional probabilities

$$\mathbb{P}\left[i_2 = i | i_1\right] \propto \mathrm{dist}^2\left(\varphi_i, \mathrm{Span}\left\{\varphi_{i_1}\right\}\right)$$

Exact sampling: 2D visualisation



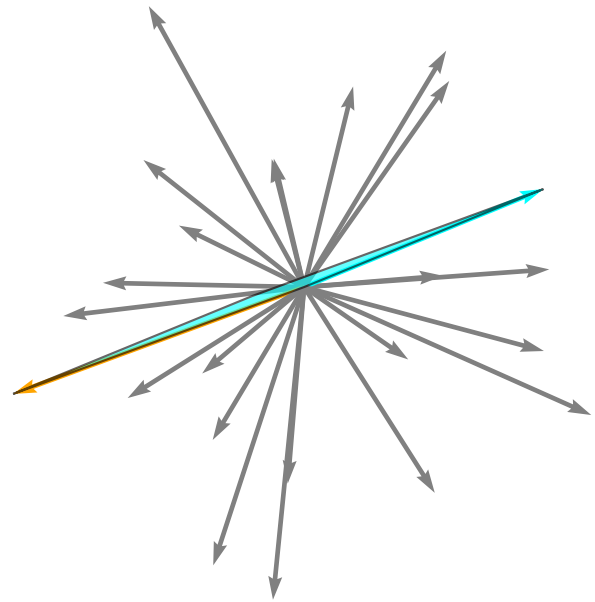


Figure 5: φ_i spanning smallest $\mathrm{Vol}^2\{\cdot, \varphi_{i_1}\}$

$$\mathbb{P}[i_2 = i | i_1] \propto \mathrm{dist}^2(\varphi_i, \mathrm{Span}\{\varphi_{i_1}\})$$

Exact sampling: 2D visualisation

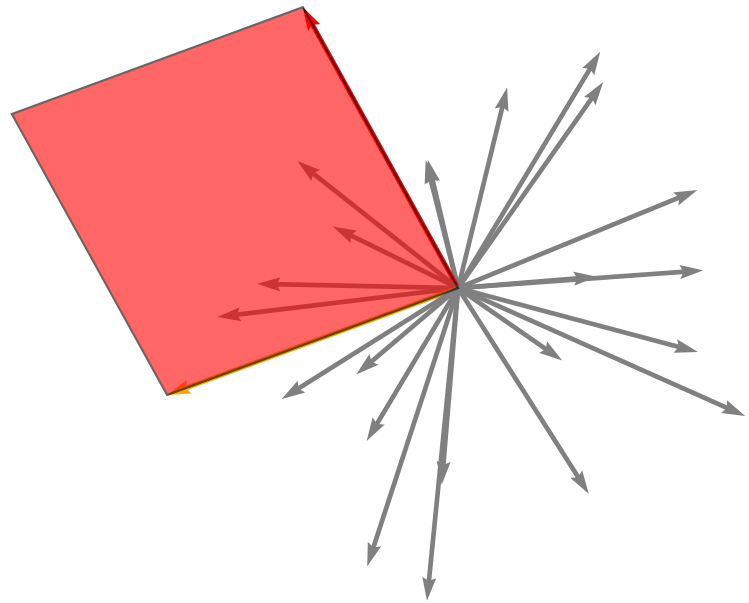
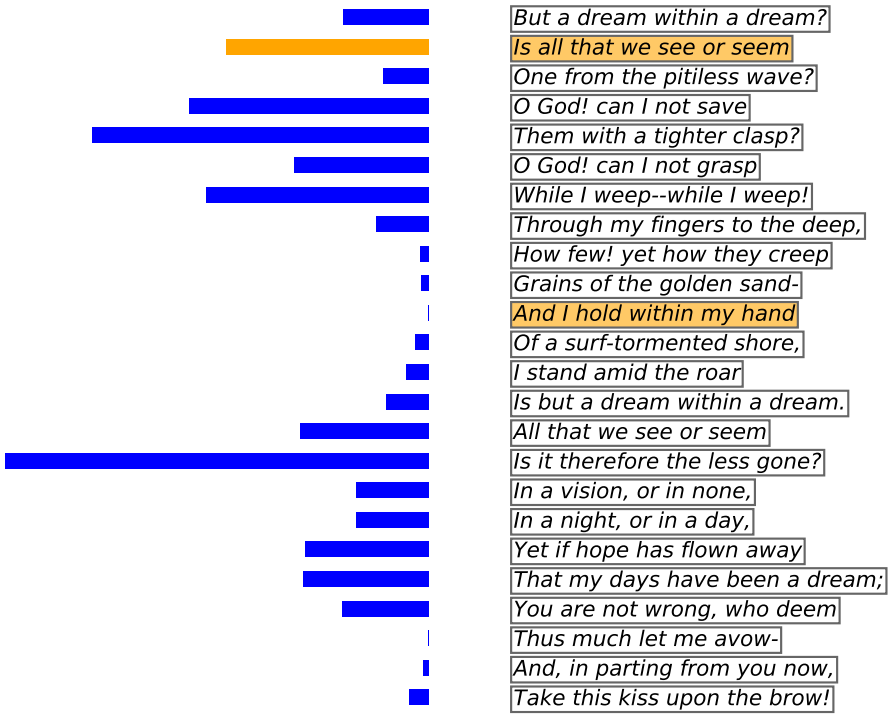


Figure 5: φ_i spanning largest $\mathrm{Vol}^2\{\cdot, \varphi_{i_1}\}$

$$\mathbb{P}[i_2 = i | i_1] \propto \mathrm{dist}^2(\varphi_i, \mathrm{Span}\{\varphi_{i_1}\})$$

Exact sampling: 2D visualisation



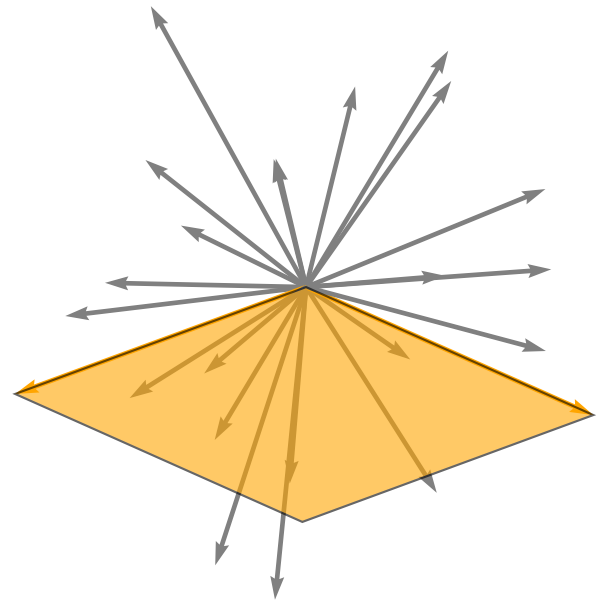


Figure 5: Feature vector selected

$$\mathbb{P}\left[i_2 = i | i_1\right] \propto \text{dist}^2\left(\varphi_i, \text{Span}\left\{\varphi_{i_1}\right\}\right)$$

Exact sampling: 2D visualisation

But a dream within a dream?
Is all that we see or seem
One from the pitiless wave?
O God! can I not save
Them with a tighter clasp?
O God! can I not grasp
While I weep--while I weep!
Through my fingers to the deep,
How few! yet how they creep
Grains of the golden sand-
And I hold within my hand
Of a surf-tormented shore,
I stand amid the roar
Is but a dream within a dream.
All that we see or seem
Is it therefore the less gone?
In a vision, or in none,
In a night, or in a day,
Yet if hope has flown away
That my days have been a dream;
You are not wrong, who deem
Thus much let me avow-
And, in parting from you now,
Take this kiss upon the brow!

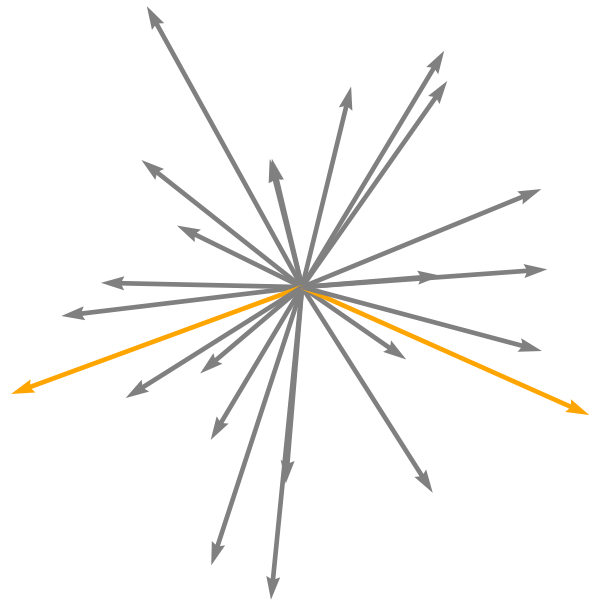
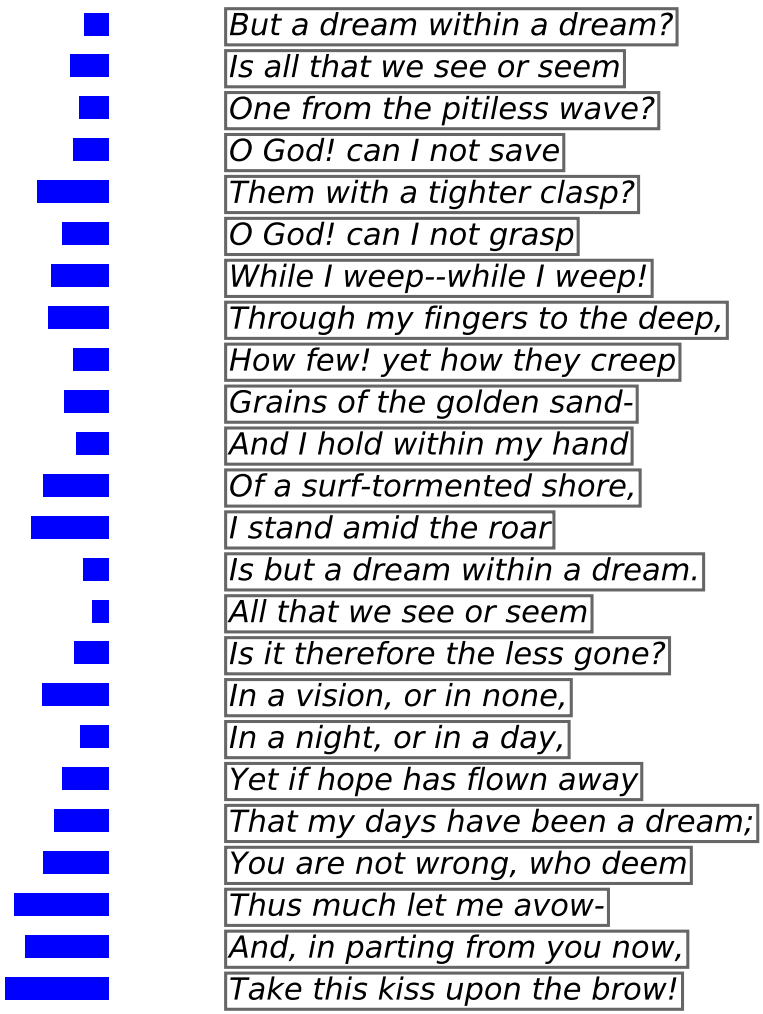


Figure 5: Summary extraction complete

$$\forall i, \qquad \operatorname{dist}^2\left(\varphi_i, \operatorname{Span}\left\{\varphi_{i_1}, \varphi_{i_2}\right\}\right) = \operatorname{dist}^2\Big(\underbrace{\varphi_i}_{\in \mathbb{R}^2}, \mathbb{R}^2\Big) = 0$$

Exact sampling in action

Consider a (orthogonal) projection matrix $\mathbf{K}$ s.t. $\operatorname{rk}\mathbf{K} = r = 10$



Exact sampling in action

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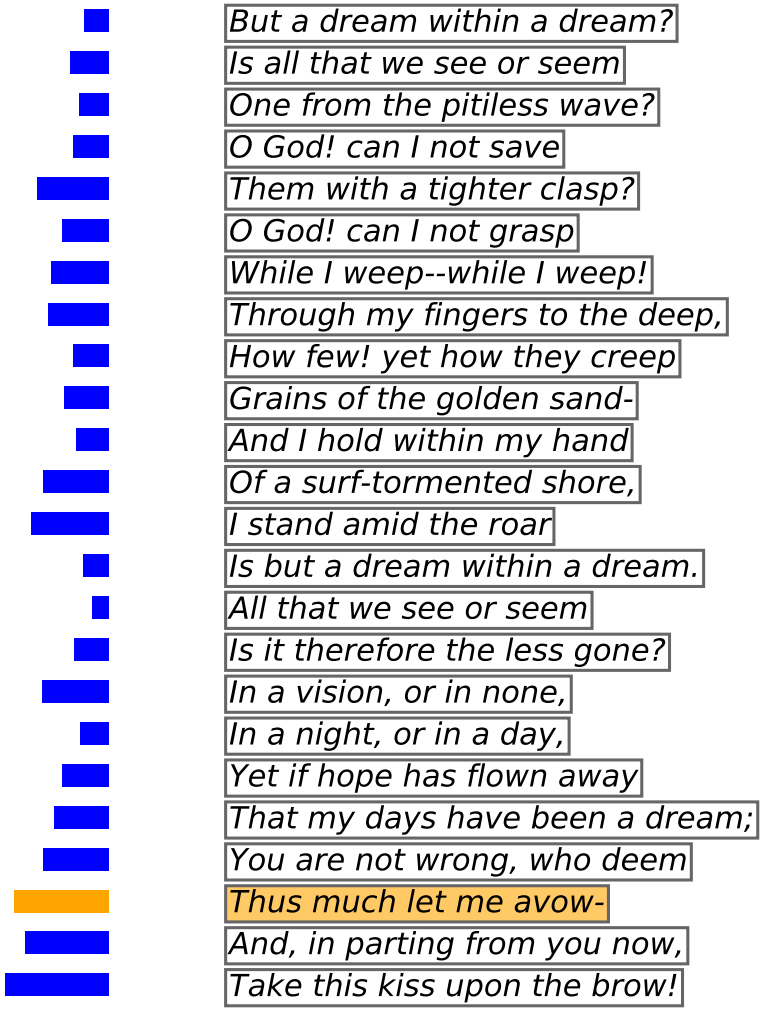

Figure 6: 1st pick

Exact sampling in action

Consider a (orthogonal) projection matrix $\mathbf{K}$ s.t. $\operatorname{rk} \mathbf{K} = r = 10$

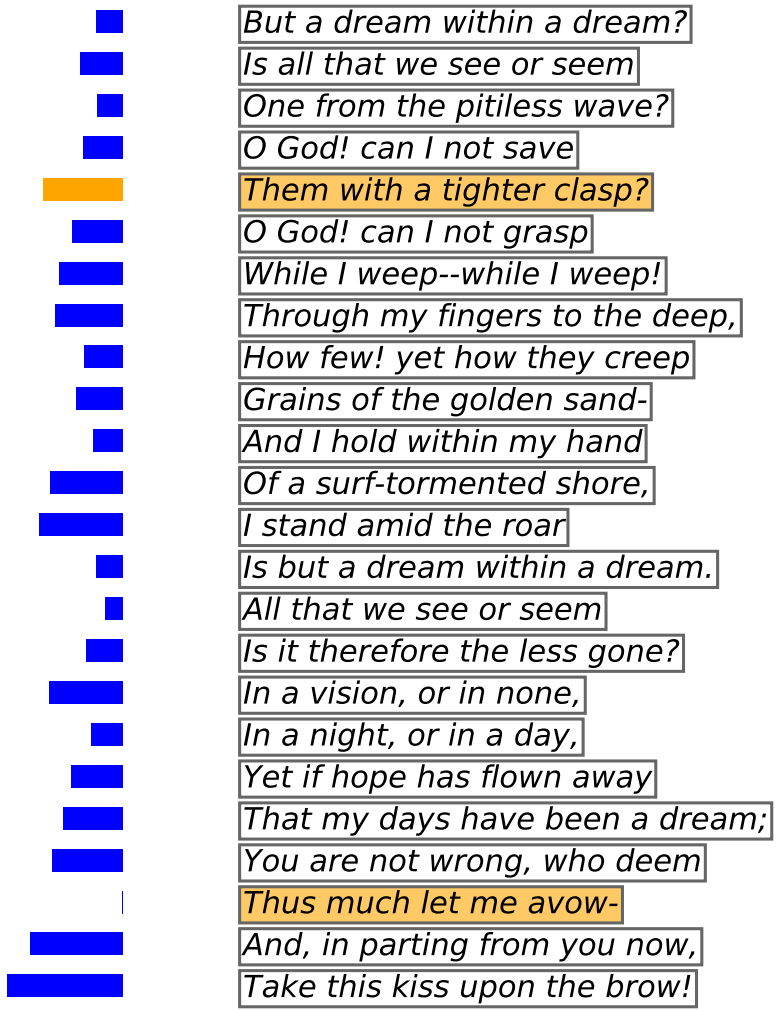

Figure 6: 2nd pick

Exact sampling in action

Consider a (orthogonal) projection matrix $\mathbf{K}$ s.t. $\operatorname{rk} \mathbf{K} = r = 10$

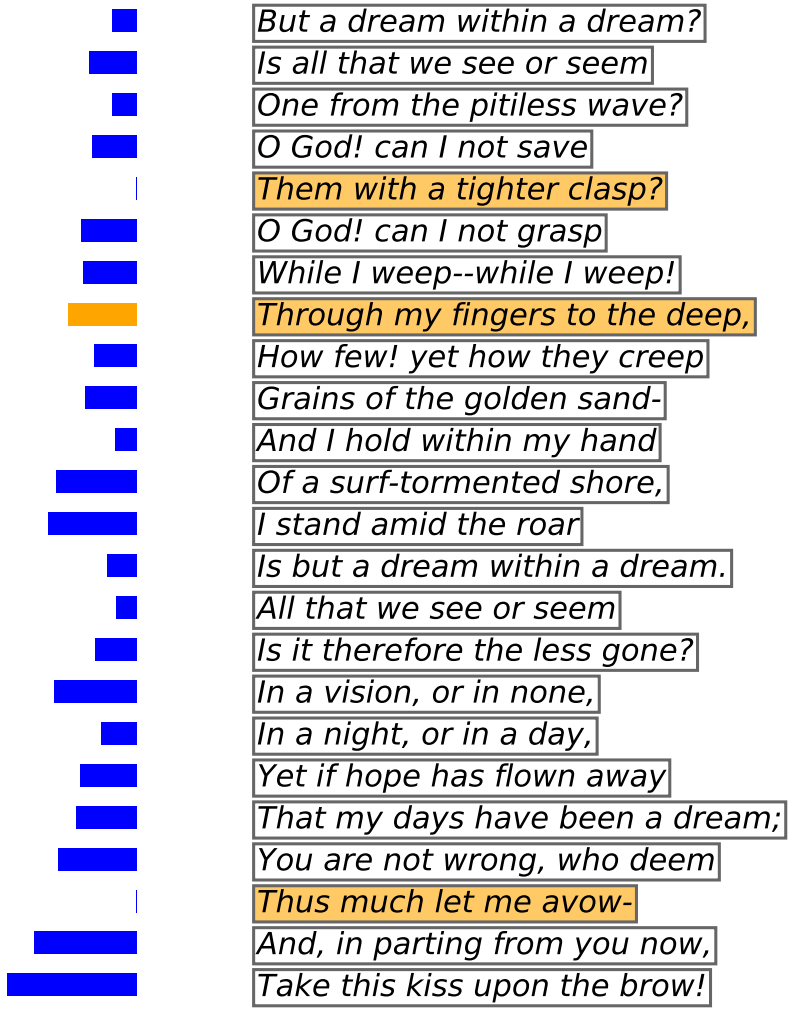

Figure 6: 3rd pick

Exact sampling in action

Consider a (orthogonal) projection matrix $\mathbf{K}$ s.t. $\operatorname{rk} \mathbf{K} = r = 10$

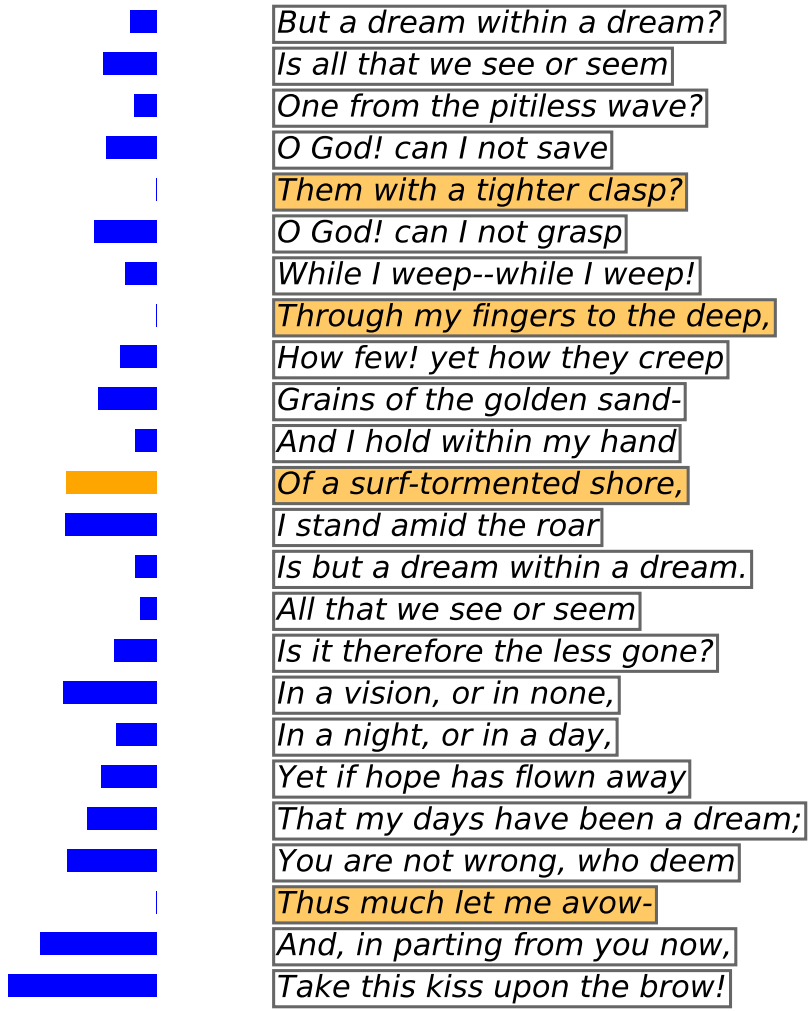

Figure 6: 4th pick

Exact sampling in action

Consider a (orthogonal) projection matrix $\mathbf{K}$ s.t. $\operatorname{rk} \mathbf{K} = r = 10$

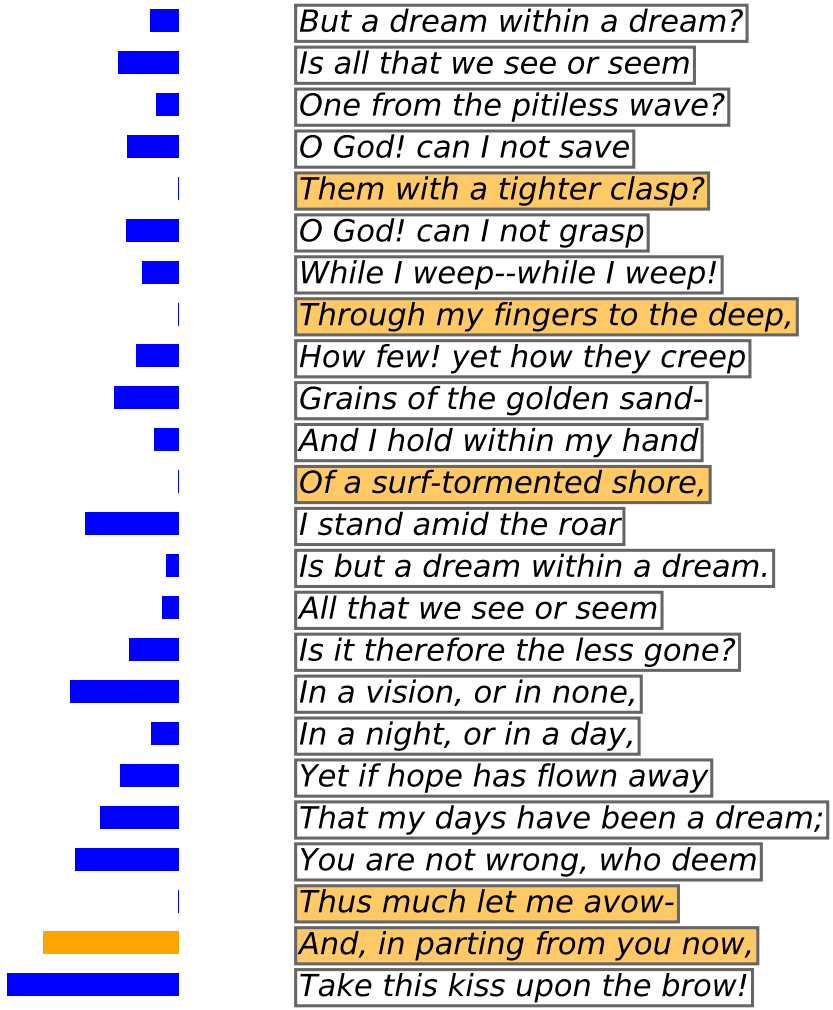

Figure 6: 5th pick

Exact sampling in action

Consider a (orthogonal) projection matrix $\mathbf{K}$ s.t. $\operatorname{rk} \mathbf{K} = r = 10$

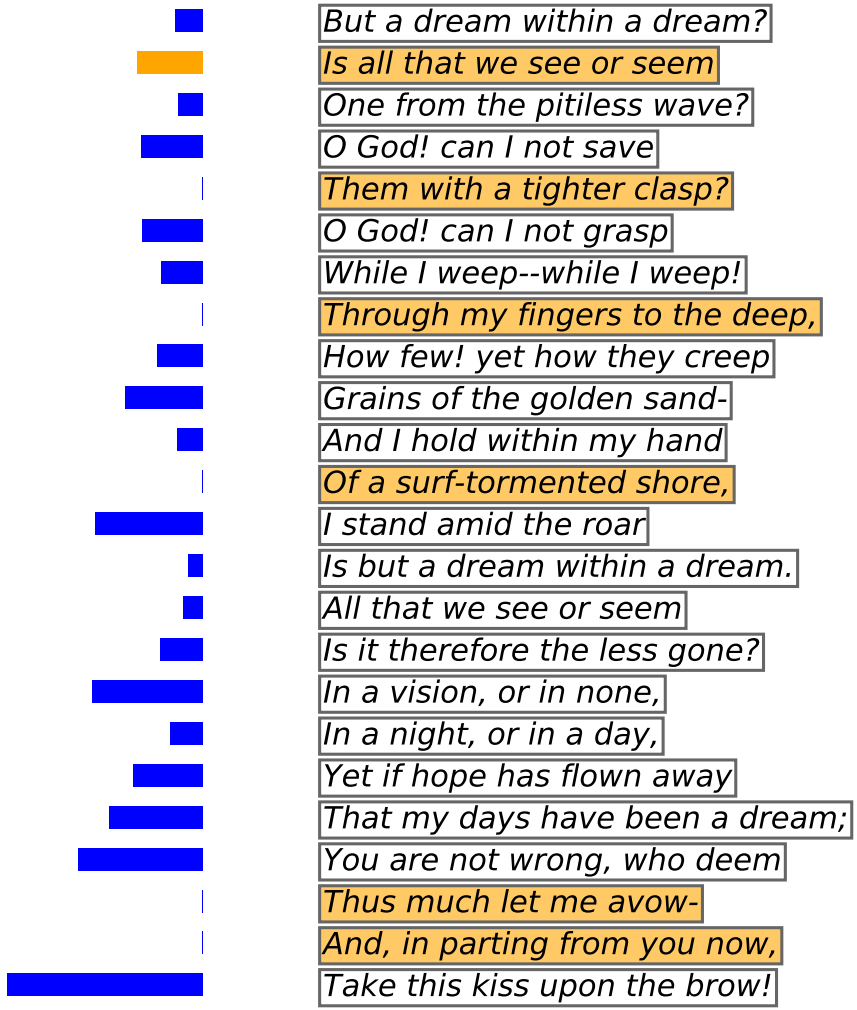

Figure 6: 6th pick

Exact sampling in action

Consider a (orthogonal) projection matrix $\mathbf{K}$ s.t. $\operatorname{rk} \mathbf{K} = r = 10$

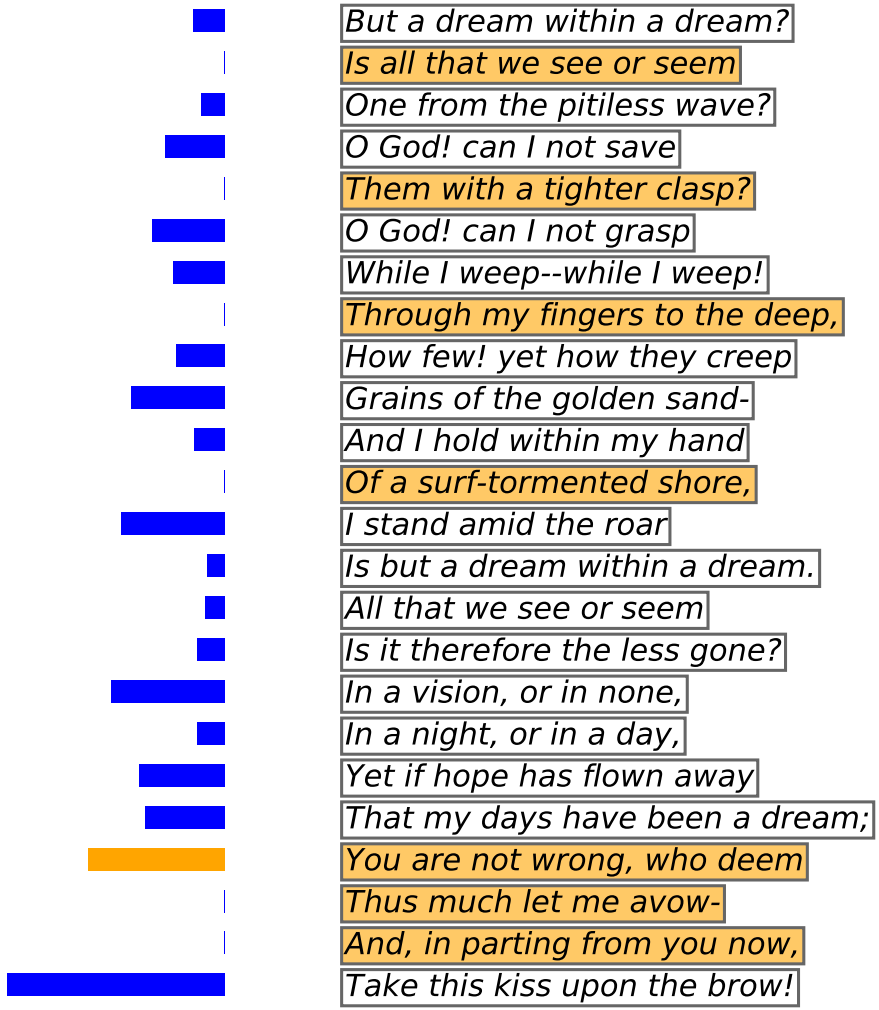

Figure 6: 7th pick

Exact sampling in action

Consider a (orthogonal) projection matrix $\mathbf{K}$ s.t. $\operatorname{rk} \mathbf{K} = r = 10$

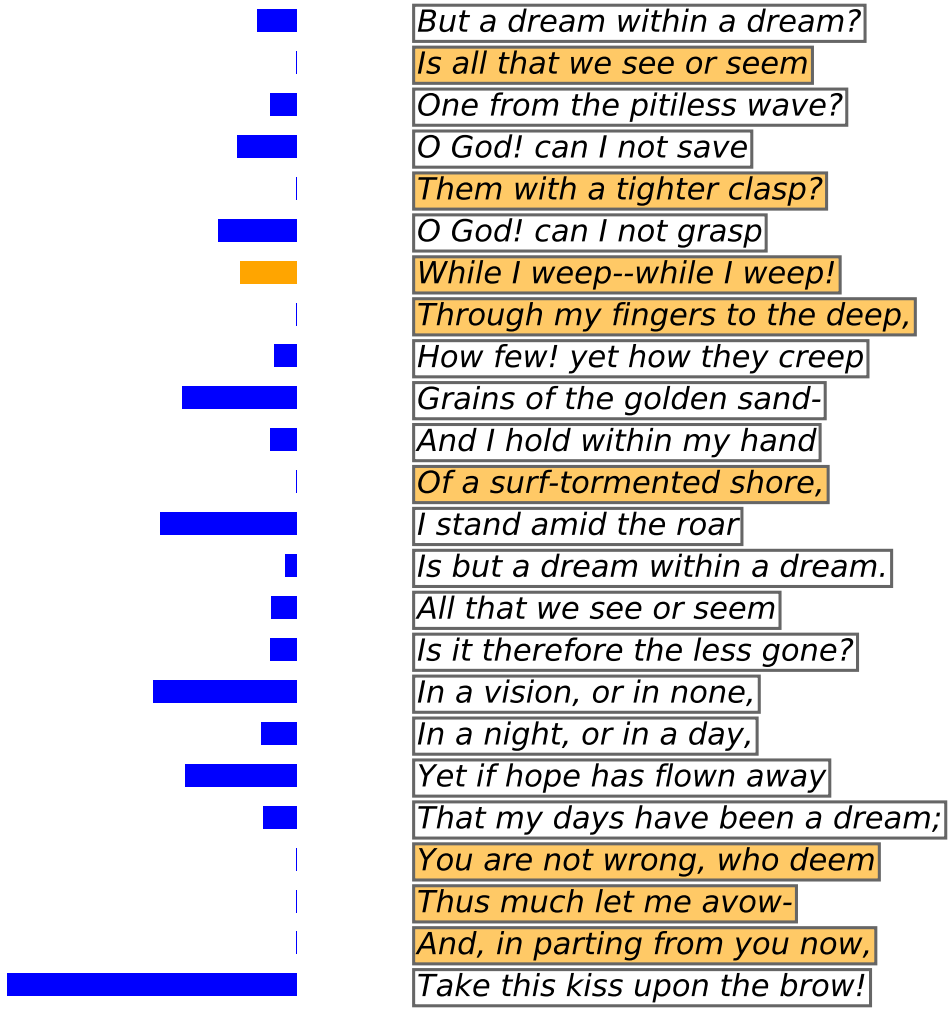

Figure 6: 8th pick

Exact sampling in action

Consider a (orthogonal) projection matrix $\mathbf{K}$ s.t. $\operatorname{rk} \mathbf{K} = r = 10$

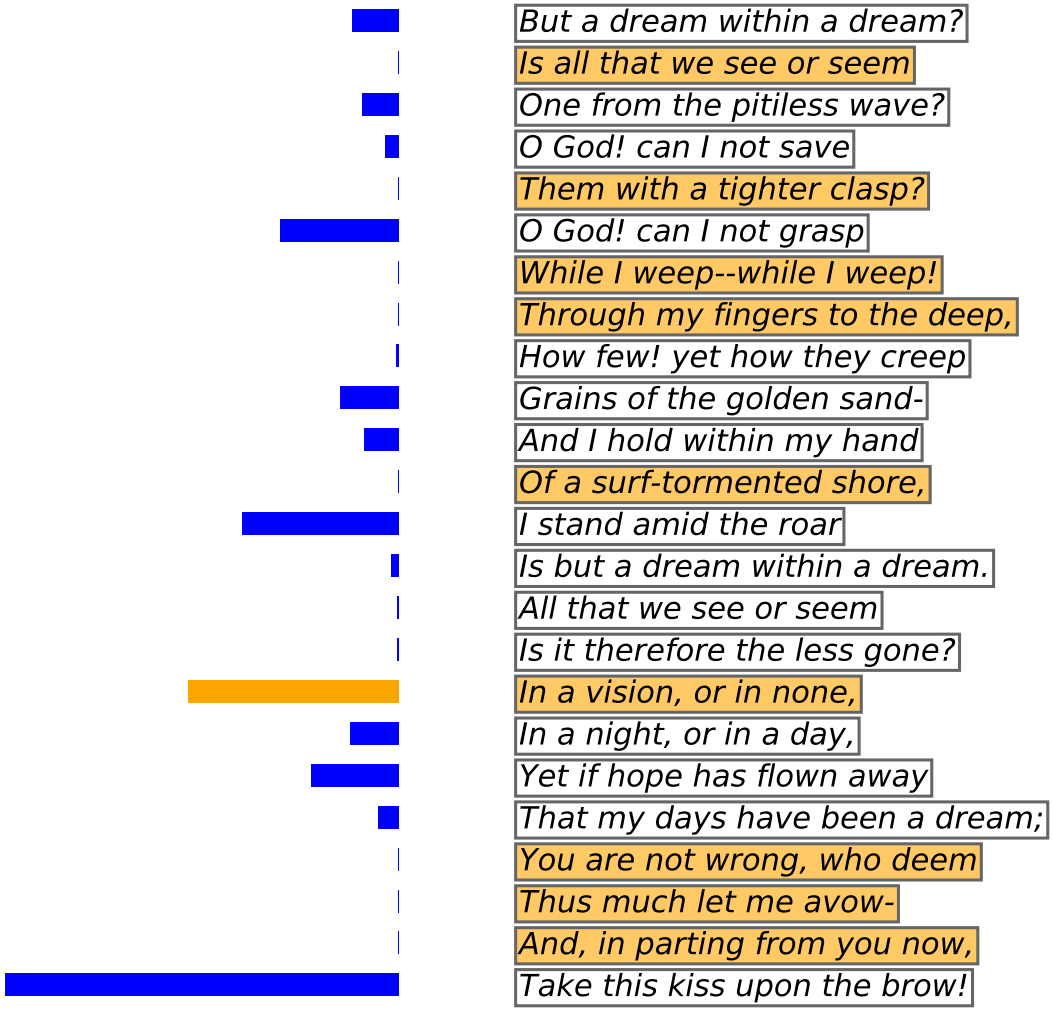

Figure 6: 9th pick

Exact sampling in action

Consider a (orthogonal) projection matrix $\mathbf{K}$ s.t. $\operatorname{rk} \mathbf{K} = r = 10$

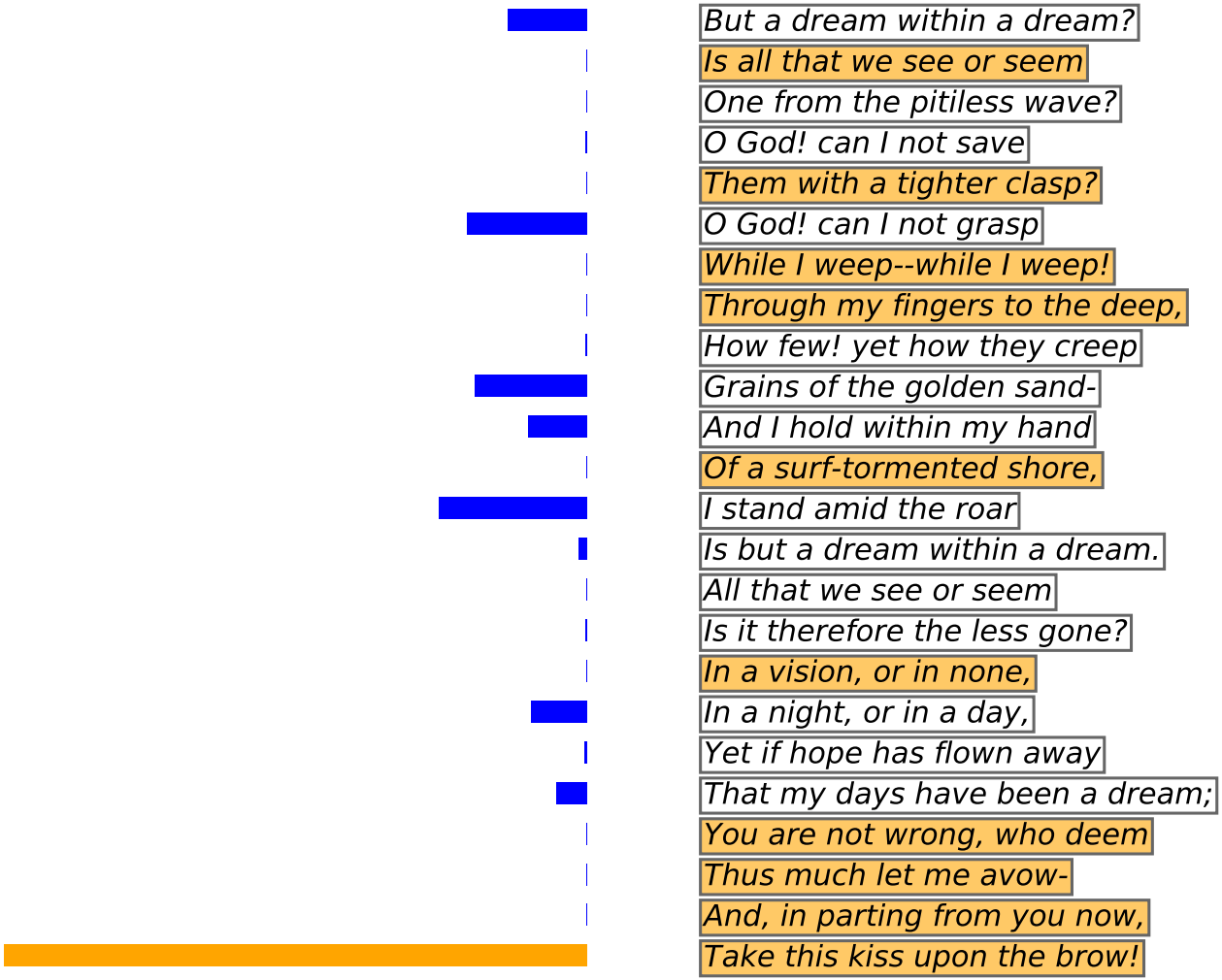

Figure 6: 10th pick

Exact sampling in action

Consider a (orthogonal) projection matrix $\mathbf{K}$ s.t. $\operatorname{rk} \mathbf{K} = r = 10$

But a dream within a dream?
Is all that we see or seem
One from the pitiless wave?
O God! can I not save
Them with a tighter clasp?
O God! can I not grasp
While I weep--while I weep!
Through my fingers to the deep,
How few! yet how they creep
Grains of the golden sand-
And I hold within my hand
Of a surf-tormented shore,
I stand amid the roar
Is but a dream within a dream.
All that we see or seem
Is it therefore the less gone?
In a vision, or in none,
In a night, or in a day,
Yet if hope has flown away
That my days have been a dream;
You are not wrong, who deem
Thus much let me avow-
And, in parting from you now,
Take this kiss upon the brow!

Figure 6: Final summary

From exact to approximate sampling

Projection kernel

$$\mathbf{K} = \mathbf{A}^\top \left[\mathbf{A}\mathbf{A}^\top\right]^{-1} \mathbf{A} = \mathbf{\Phi}^\top \mathbf{\Phi}$$

where

$$\mathbf{A} = \left(\sqrt{q_1}\phi_1 | \dots | \sqrt{q_N}\phi_N\right) \quad \text{and} \quad \mathbf{\Phi} = \left(\varphi_1 | \dots | \varphi_N\right)$$

- Exact sampling
  - Costly cf. Gram-Schmidt $= \mathcal{O}(rN^2)$
  - $\approx$ interpretable ϕ vs. φ
- Approximate sampling
  - Build a Markov chain on $\mathcal{B}$ with stationary distribution DPP($\mathbf{K}$)
  - Graphical representation in terms of the features ϕ_is
  - More interpretable

Key idea from Dyer and Frieze (1994)

Volume spanned by the feature vectors

$$\mathcal{Z}(\mathbf{A}) \triangleq \mathbf{A}[0,1]^N$$

admits a natural tiling

$$\text{Vol}\,\mathcal{Z}(\mathbf{A}) = \sum_{B\in\mathcal{B}} \text{Vol}\,\mathcal{Z}(\mathbf{A}_{:B}) = \sum_{B\in\mathcal{B}} |\det \mathbf{B}|$$

$$\begin{array}{ll} \min\limits_{y\in\mathbb{R}^N} & c_1^\mathsf{T} y \\ \text{s.t.} & \mathbf{A}y = x \\ & 0^N \le y \le 1^N \end{array}$$

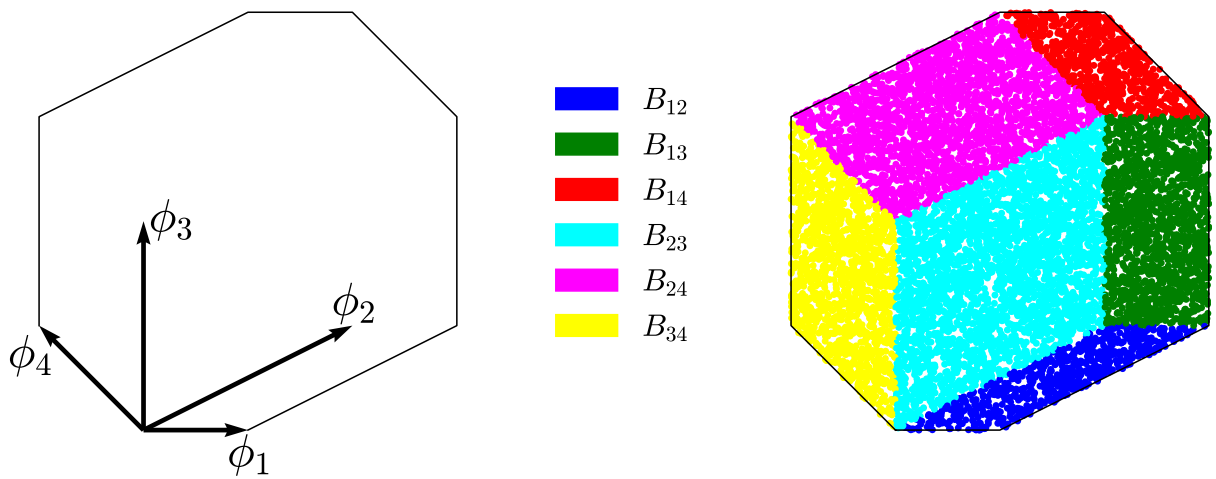

Figure 7: $\mathcal{Z}(\mathbf{A})$ and a possible tiling, $\mathbf{A} = \left(\begin{smallmatrix} 1 & 2 & 0 & -1 \\ 0 & 1 & 2 & 1 \end{smallmatrix}\right)$

Key idea from Dyer and Frieze (1994)

Volume spanned by the feature vectors

$$\mathcal{Z}(\mathbf{A}) \triangleq \mathbf{A}[0,1]^N$$

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$$\begin{array}{ll} \min\limits_{y \in \mathbb{R}^N} & c_2^\mathsf{T} y \\ \text{s.t.} & \mathbf{A}y = x \\ & 0^N \leq y \leq 1^N \end{array}$$

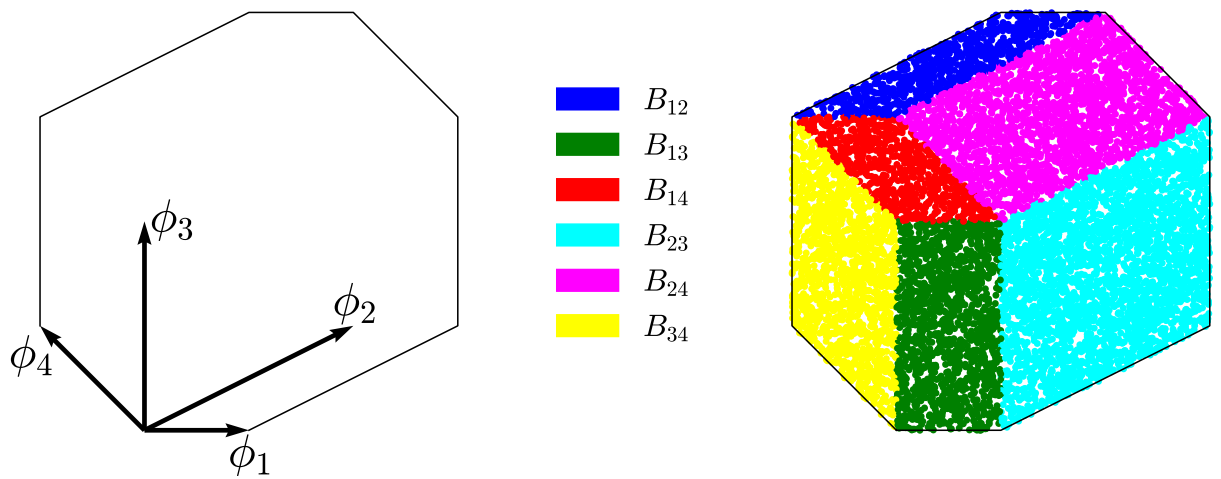


Figure 7: $\mathcal{Z}(\mathbf{A})$ and another possible tiling, $\mathbf{A} = \left(\begin{smallmatrix} 1 & 2 & 0 & -1 \\ 0 & 1 & 2 & 1 \end{smallmatrix}\right)$

Random walk on tiles - 1

Basis exchange (Feder and Mihail, 1992; Li et al., 2016)

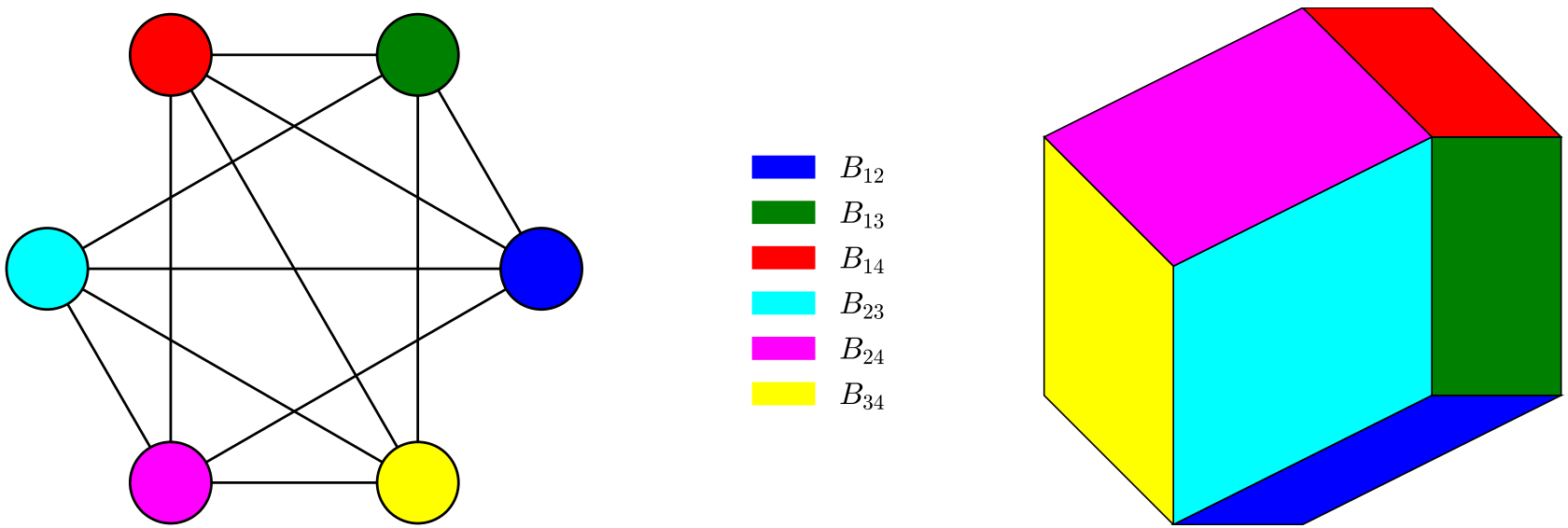


Figure 8: Random walk on tiles

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 0 & -1 \\ 0 & 1 & 2 & 1 \end{pmatrix}$$

basisExchange in action

But a dream within a dream?
Is all that we see or seem
One from the pitiless wave?
O God! can I not save
Them with a tighter clasp?
O God! can I not grasp
While I weep--while I weep!
Through my fingers to the deep,
How few! yet how they creep
Grains of the golden sand-
And I hold within my hand
Of a surf-tormented shore,
I stand amid the roar
Is but a dream within a dream.
All that we see or seem
Is it therefore the less gone?
In a vision, or in none,
In a night, or in a day,
Yet if hope has flown away
That my days have been a dream;
You are not wrong, who deem
Thus much let me avow-
And, in parting from you now,
Take this kiss upon the brow!

Figure 9: Iteration 0

basisExchange in action

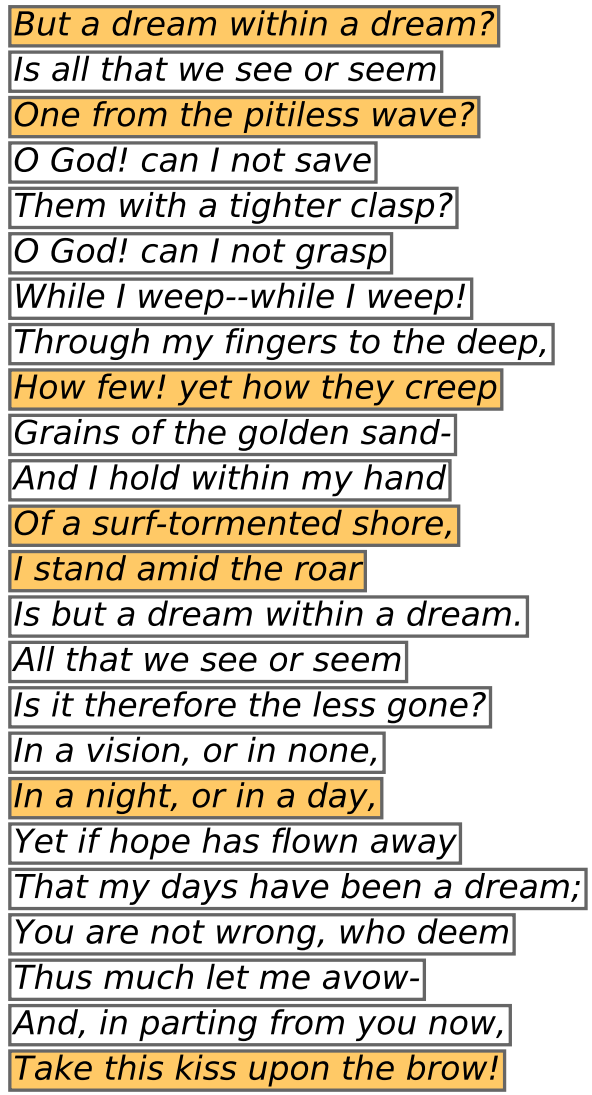


Figure 9: Iteration 1

basisExchange in action

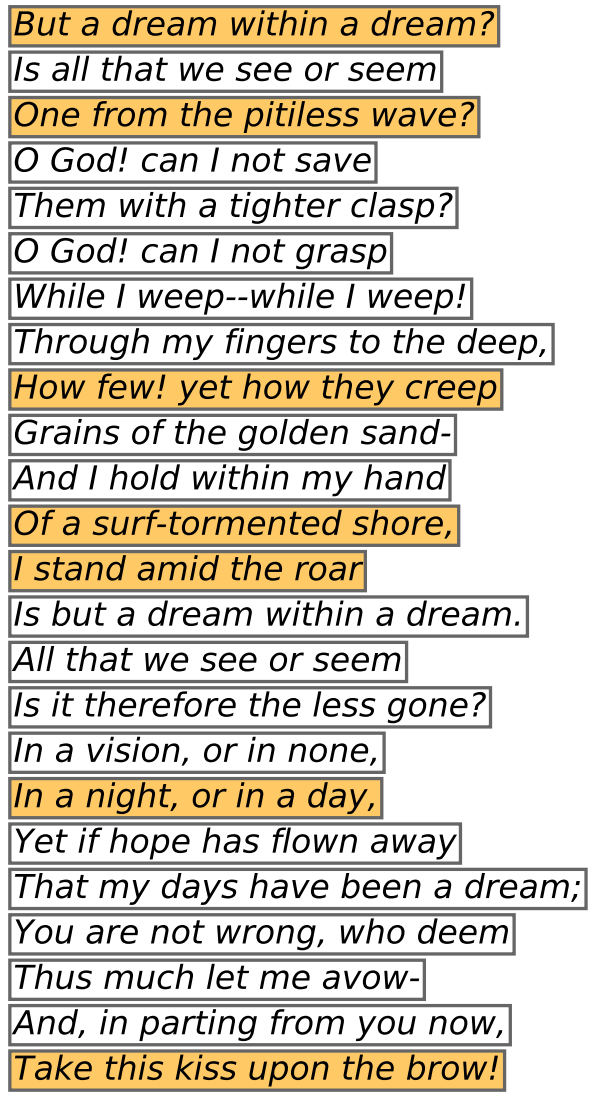


Figure 9: Iteration 2

basisExchange in action

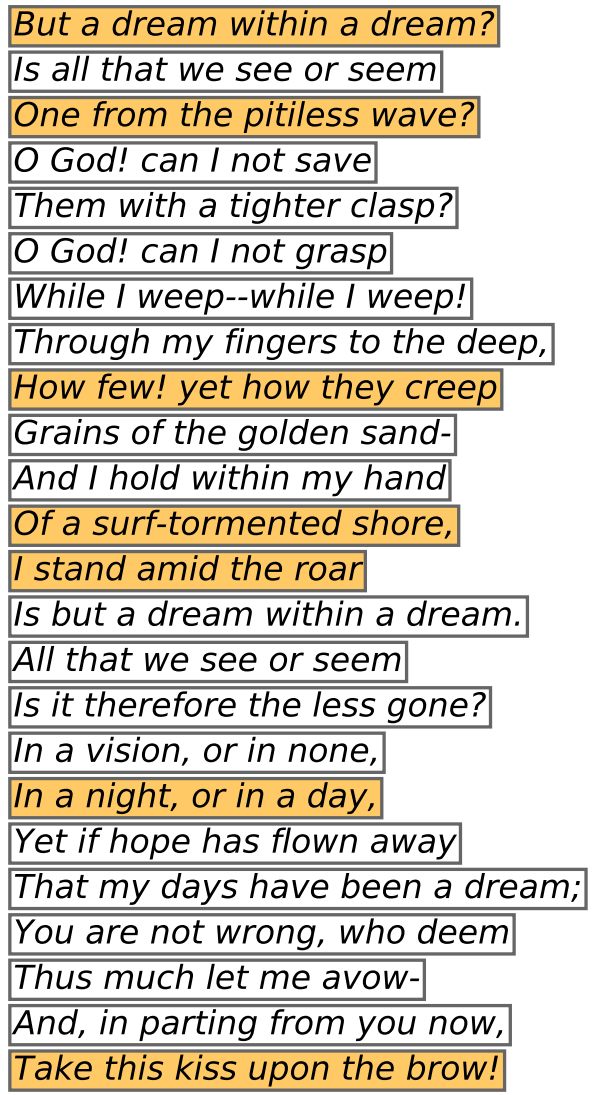


Figure 9: Iteration 3

basisExchange in action

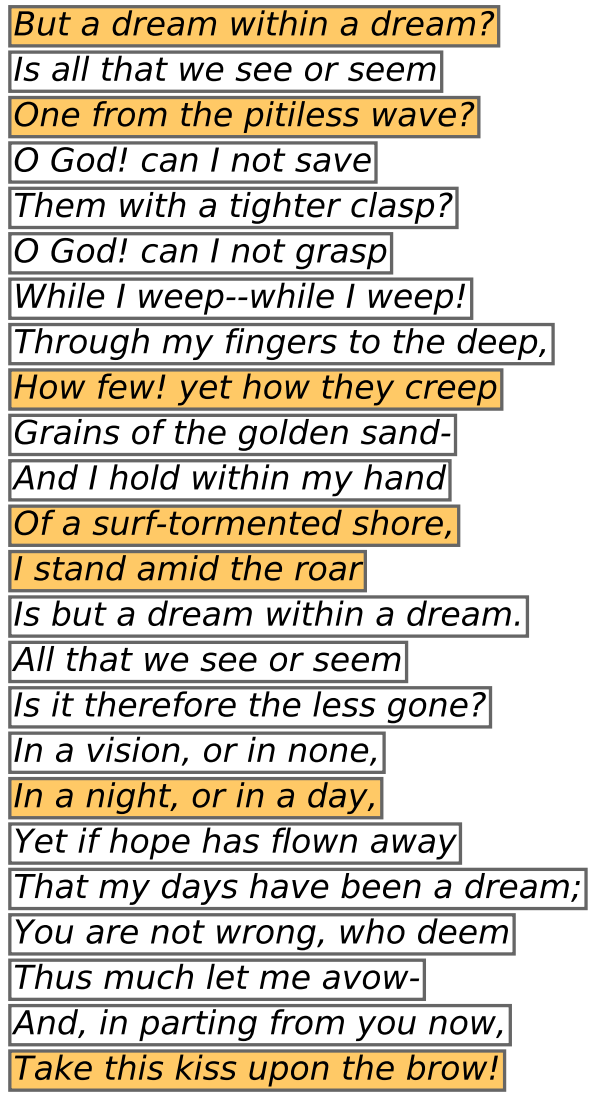


Figure 9: Iteration 4

basisExchange in action

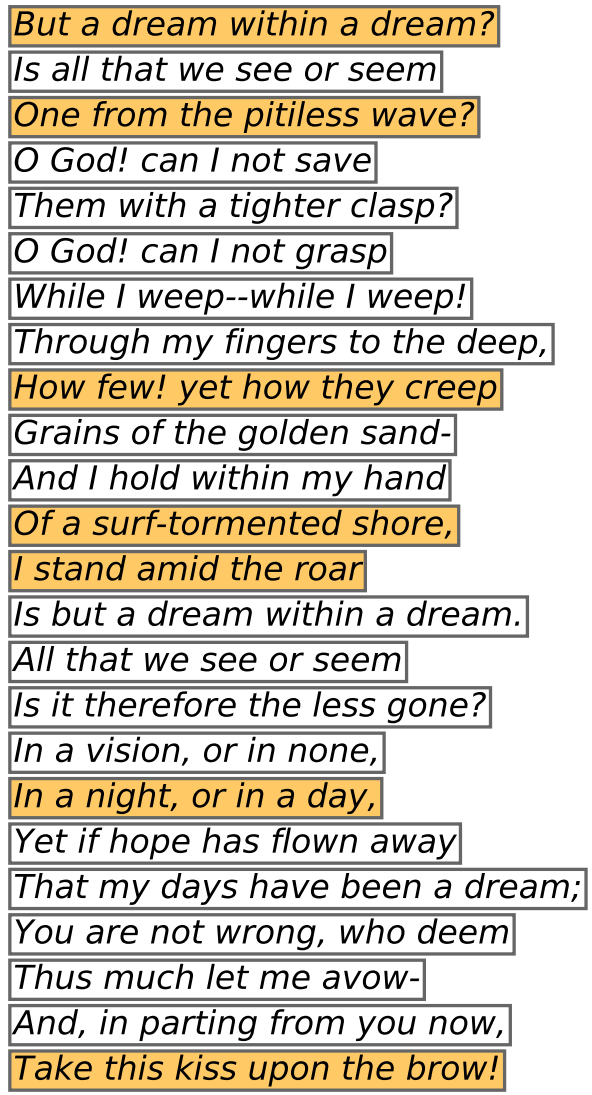


Figure 9: Iteration 5

basisExchange in action

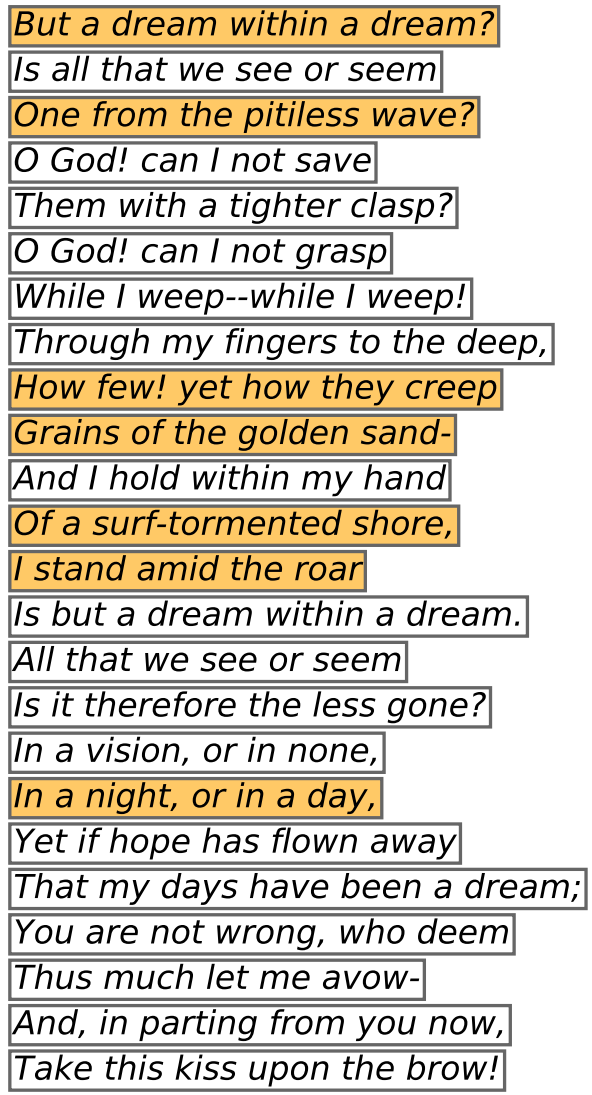


Figure 9: Iteration 6

basisExchange in action

But a dream within a dream?
Is all that we see or seem
One from the pitiless wave?
O God! can I not save
Them with a tighter clasp?
O God! can I not grasp
While I weep--while I weep!
Through my fingers to the deep,
How few! yet how they creep
Grains of the golden sand-
And I hold within my hand
Of a surf-tormented shore,
I stand amid the roar
Is but a dream within a dream.
All that we see or seem
Is it therefore the less gone?
In a vision, or in none,
In a night, or in a day,
Yet if hope has flown away
That my days have been a dream;
You are not wrong, who deem
Thus much let me avow-
And, in parting from you now,
Take this kiss upon the brow!

Figure 9: Iteration 7

basisExchange in action

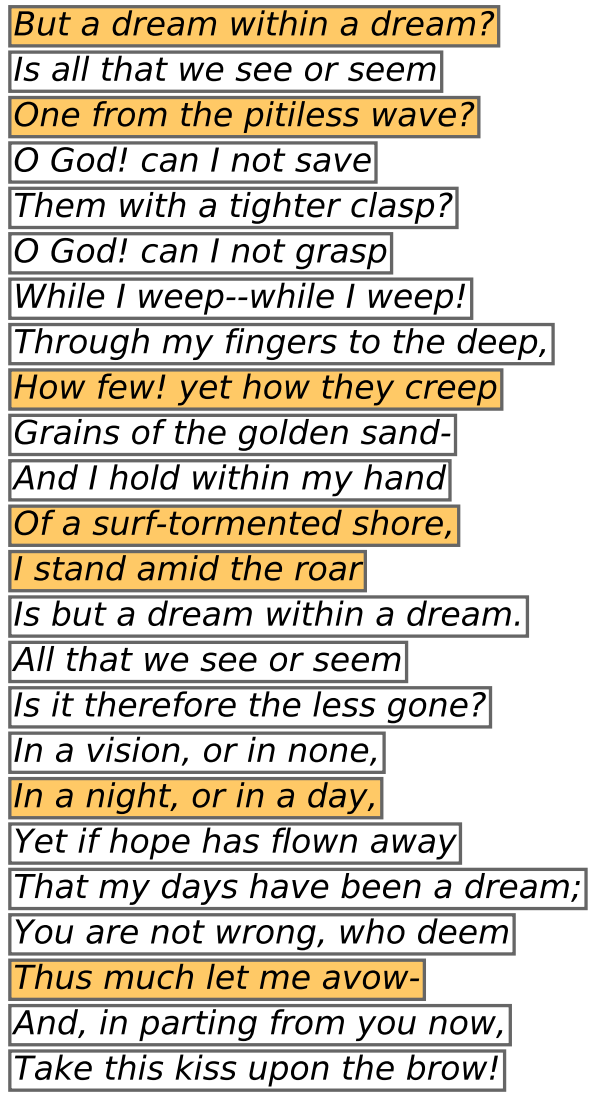


Figure 9: Iteration 8

basisExchange in action

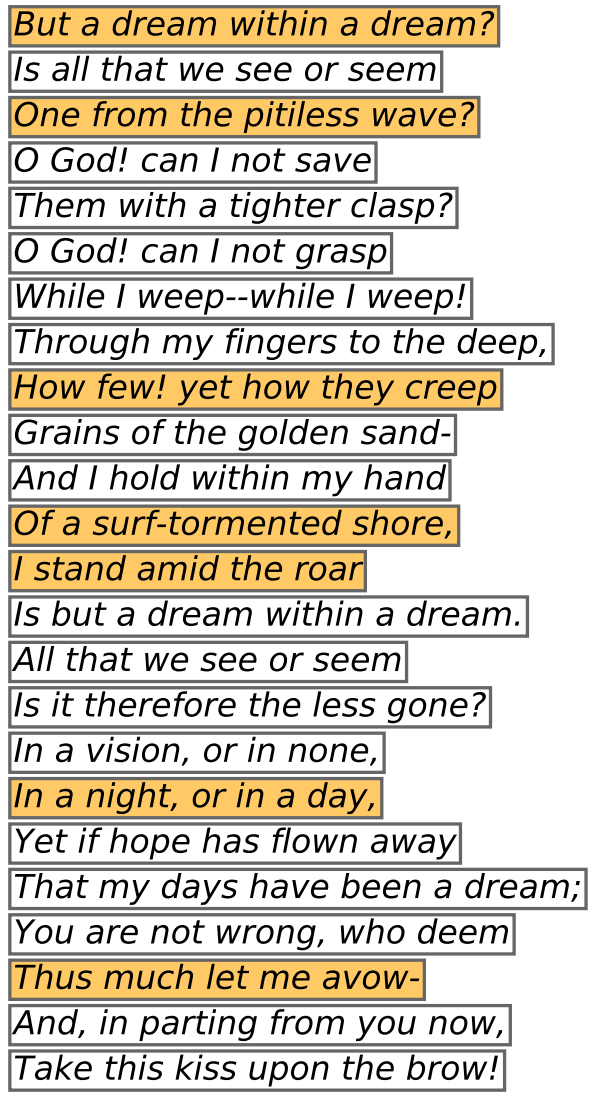


Figure 9: Iteration 9

basisExchange in action

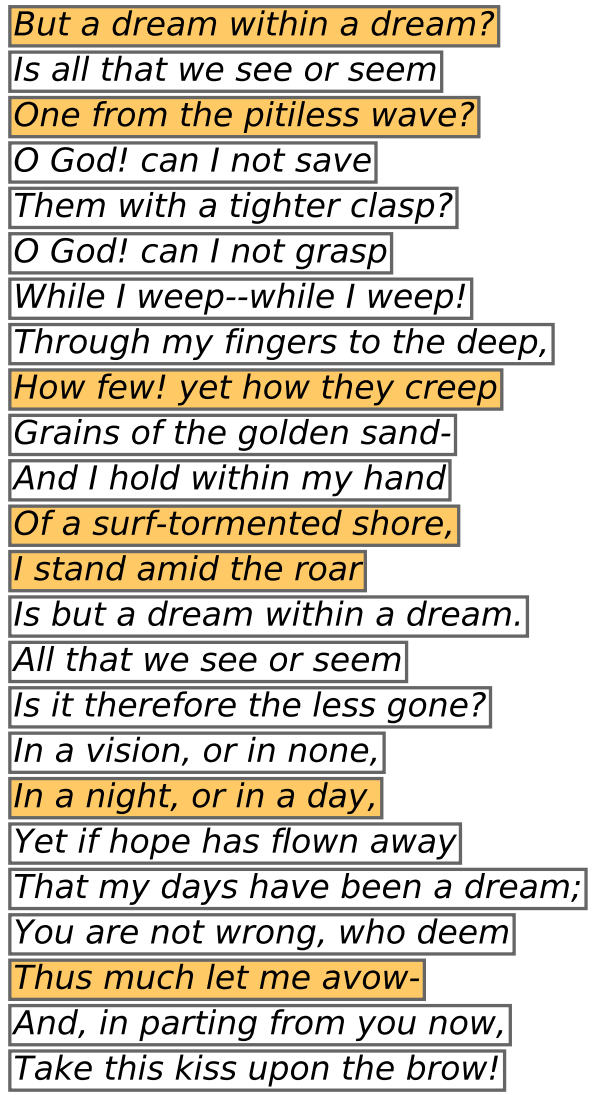
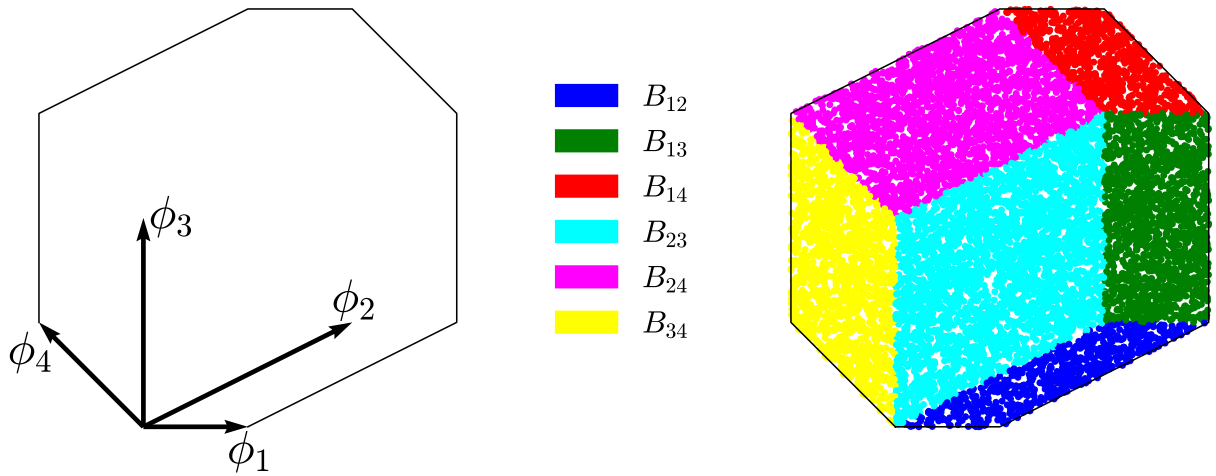


Figure 9: Iteration 10

From basisExchange to zonoSampling

- Basis exchange performs local moves.
- How to make more decorrelated moves? (Gautier et al., 2017)
  - Build a Markov chain on $\mathcal{Z}(\mathbf{A})$ that leaves invariant a specific measure $\pi(x)\,\mathrm{d}x$



  - Identify the tile in which x falls at each iteration using

$$\begin{aligned} \min_{y \in \mathbb{R}^N} \quad & c^\mathsf{T} y \\ \text{s.t.} \quad & \mathbf{A}y = x \\ & 0^N \leq y \leq 1^N \end{aligned}$$

  - Get a Markov chain on tiles that leaves DPP($\mathbf{K}$) invariant

Random walk on tiles - 2

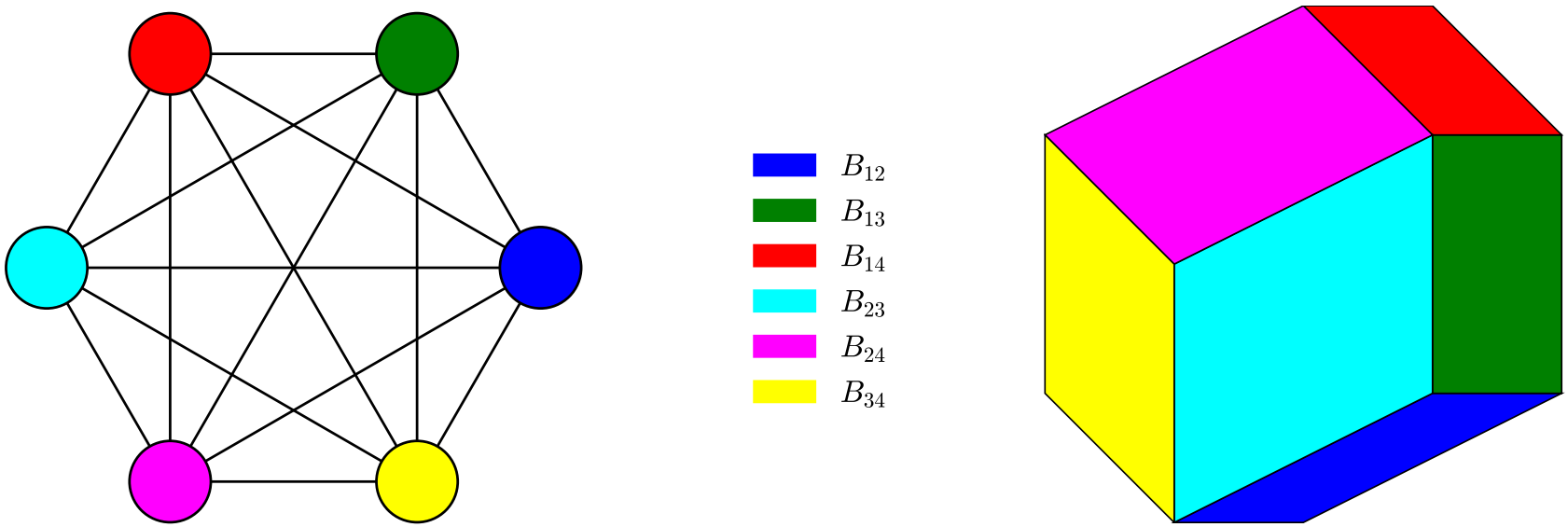


Figure 10: Random walk on tiles

- Continuous random walk in $\mathcal{Z}(\mathbf{A})$ with limiting distribution

$$\pi(x)\,\mathrm{d}x = \sum_{B\in\mathcal{B}} C_{\mathbf{B}} \times \mathbb{1}_{\mathbf{B}}(x)\,\mathrm{d}x = \sum_{ij}^{\neq} C_{\mathbf{B}_{ij}} \times \mathbb{1}_{\mathbf{B}_{ij}}(x)\,\mathrm{d}x$$

- Identification of the tile in which x falls

$$\mathbb{P}\left[x \in \mathbf{B}_{ij}\right] \propto \int_{\mathbf{B}_{ij}} \pi(x)\,\mathrm{d}x = C_{\mathbf{B}_{ij}} \times \mathrm{Vol}\,\mathbf{B}_{ij}$$

Underlying continuous random walk

To wander in convex bodies use hit-and-run (Cousins and Vempala, 2016; Lovász and Vempala, 2003)

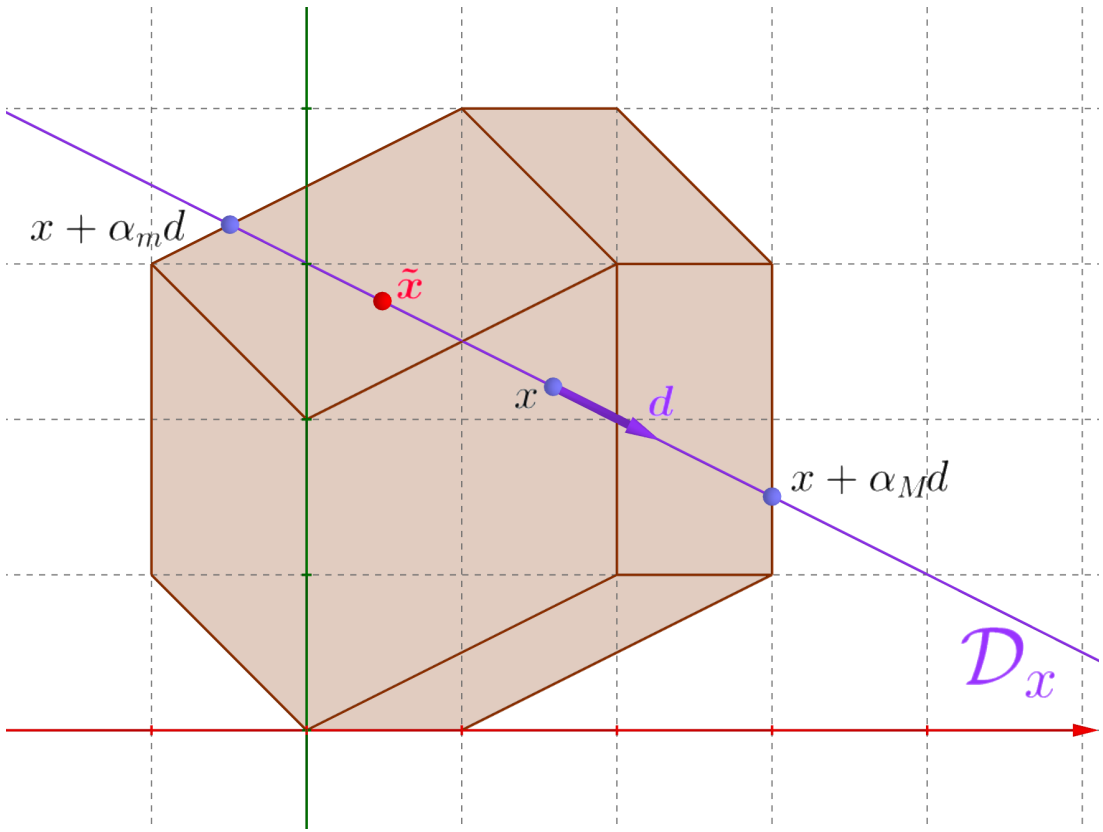
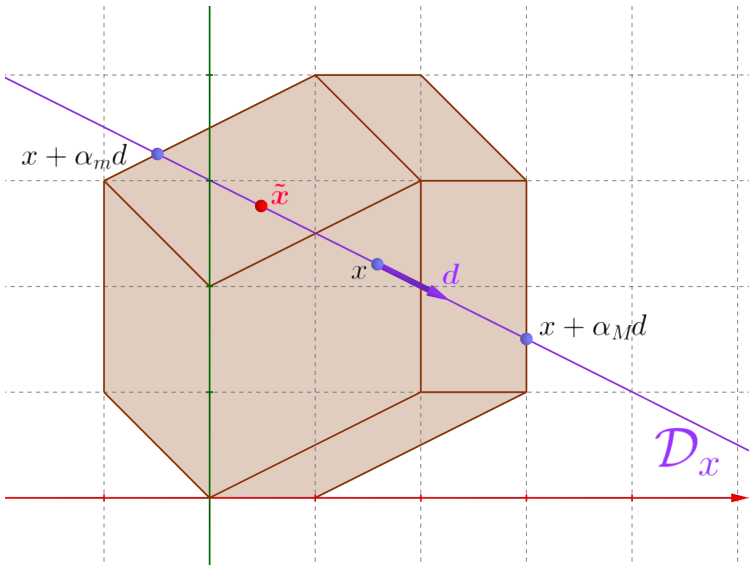


Figure 11: Hit-and-run procedure

Acceptance = 1

yields uniform target

$$\pi(x)\,\mathrm{d}x = \mathbb{1}_{\mathcal{Z}(\mathbf{A})}(x)\,\mathrm{d}x = \sum_{i,j}^{\neq} 1 \times \mathbb{1}_{\mathbf{B}_{ij}}(x)\,\mathrm{d}x$$



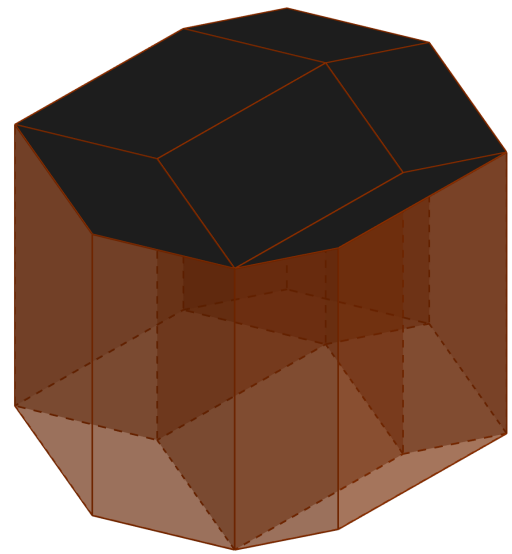


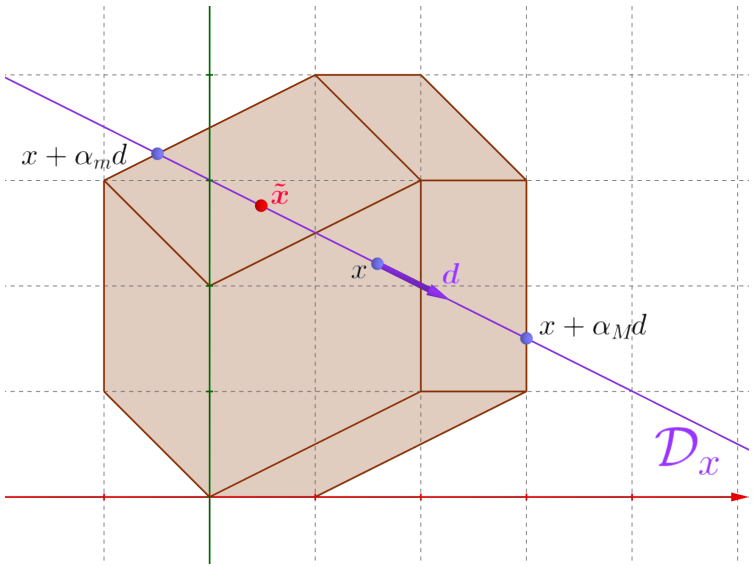
Figure 12: Uniform target

$$\mathbb{P}\left[x \in \mathbf{B}_{ij}\right] \propto \int_{\mathbf{B}_{ij}} \pi(x)\,\mathrm{d}x = 1 \times \mathrm{Vol}^{1}\,\mathbf{B}_{ij}$$

$\text{Acceptance} = \frac{\text{Vol}\, B(\widetilde{x})}{\text{Vol}\, B(x)}$

yields volume target

$$\pi(x)\,\mathrm{d}x = \sum_{i,j}^{\neq} \text{Vol}\,\mathbf{B}_{ij} \times \mathbb{1}_{\mathbf{B}_{ij}}(x)\,\mathrm{d}x$$



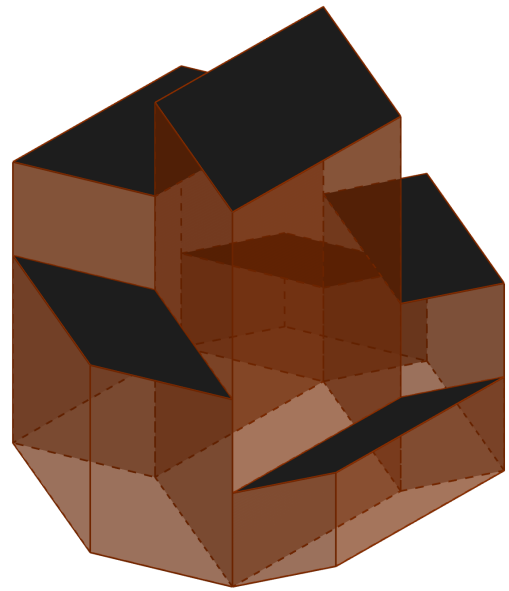


Figure 13: Volume target

$$\mathbb{P}\left[x \in \mathbf{B}_{ij}\right] \propto \int_{\mathbf{B}_{ij}} \pi(x)\,\mathrm{d}x = \text{Vol}\,\mathbf{B}_{ij}\,\text{Vol}\,\mathbf{B}_{ij} = \text{Vol}^2\,\mathbf{B}_{ij}$$

zonoSampling in action

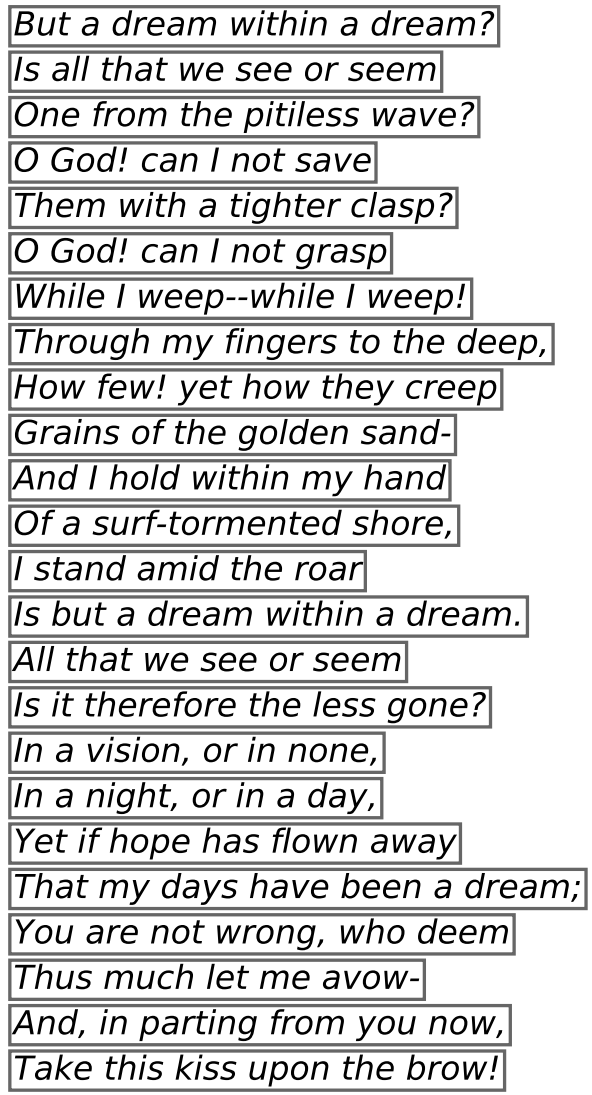


Figure 14: Iteration 0

zonoSampling in action

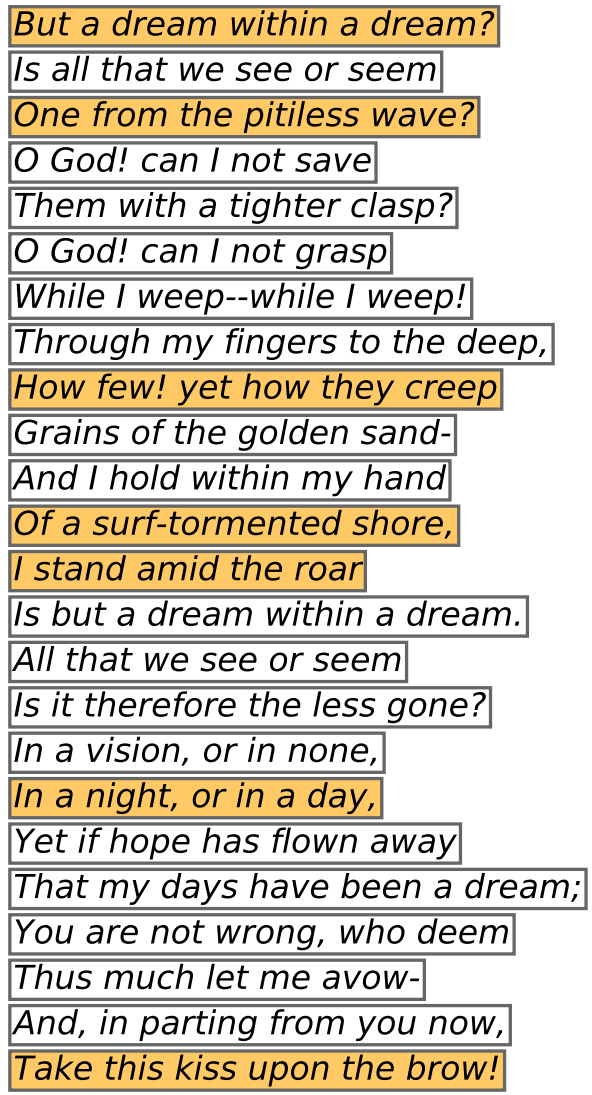


Figure 14: Iteration 1

zonoSampling in action

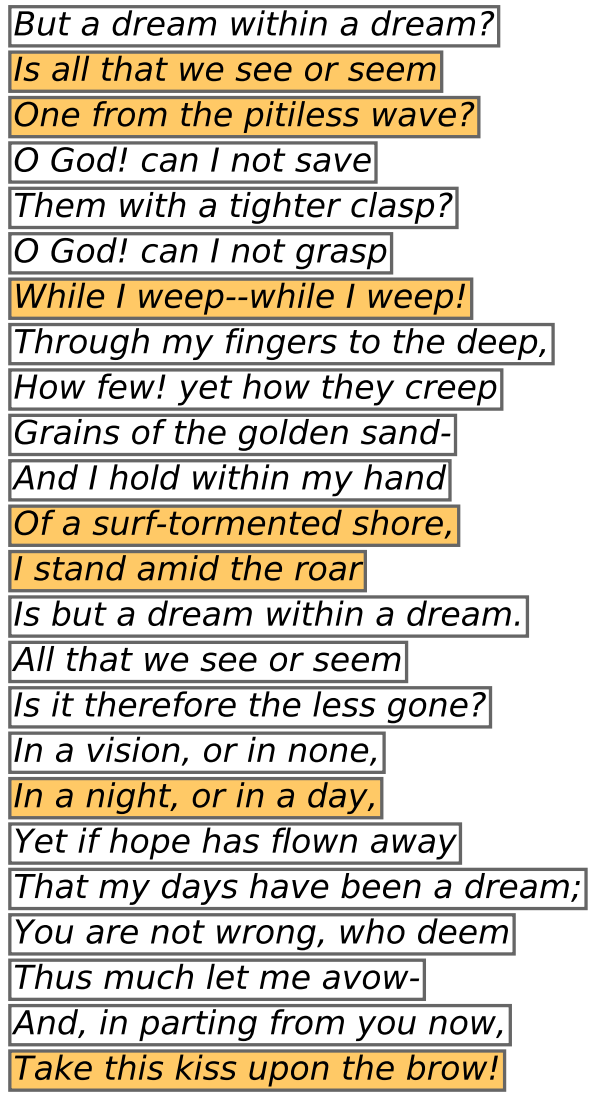


Figure 14: Iteration 2

zonoSampling in action

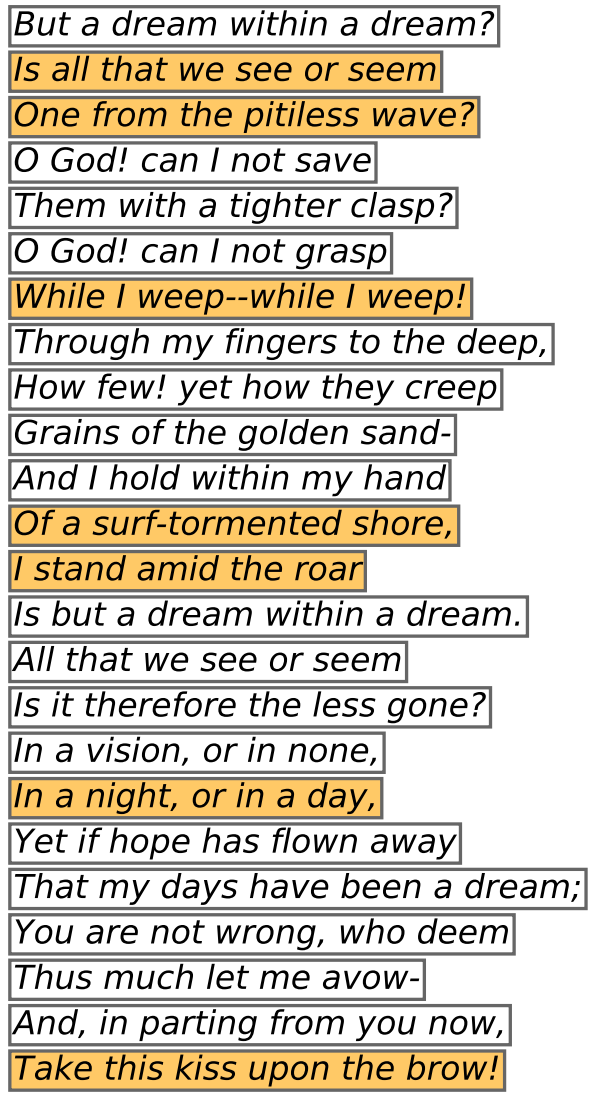


Figure 14: Iteration 3

zonoSampling in action

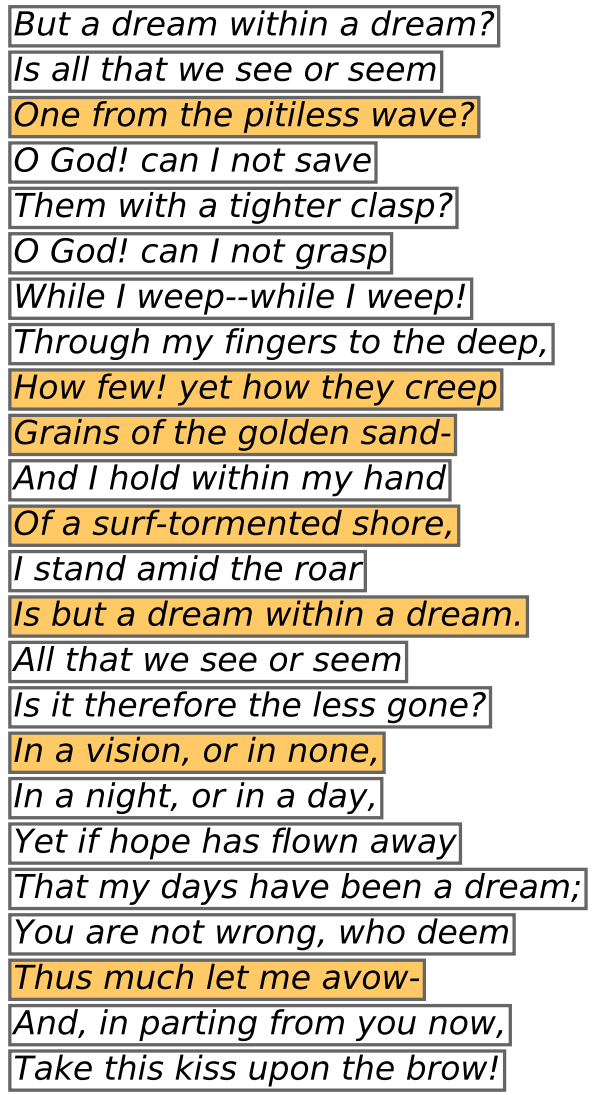


Figure 14: Iteration 4

zonoSampling in action

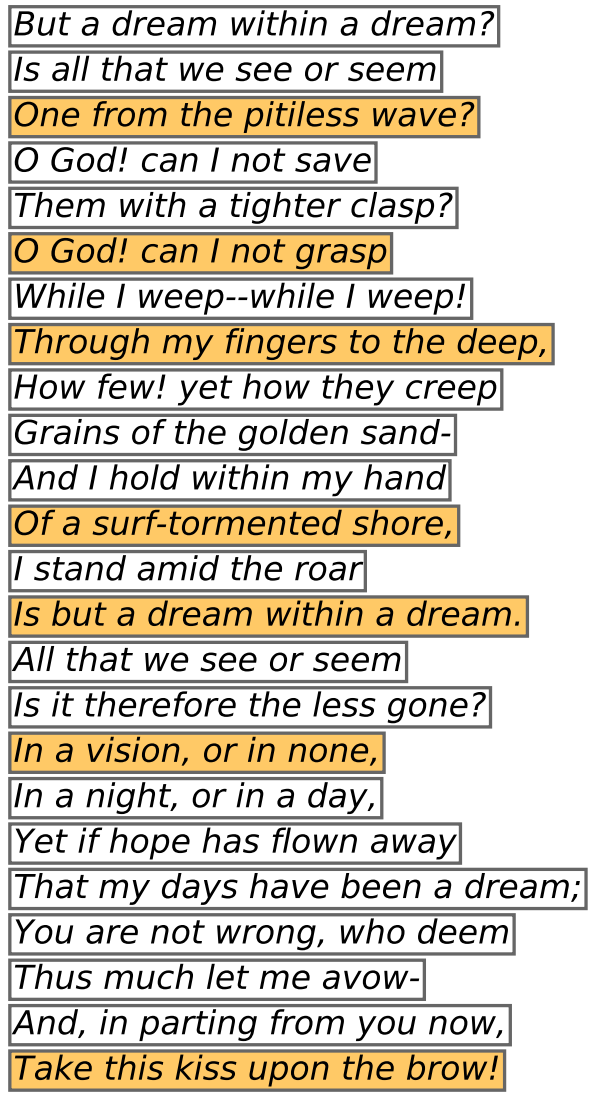


Figure 14: Iteration 5

zonoSampling in action

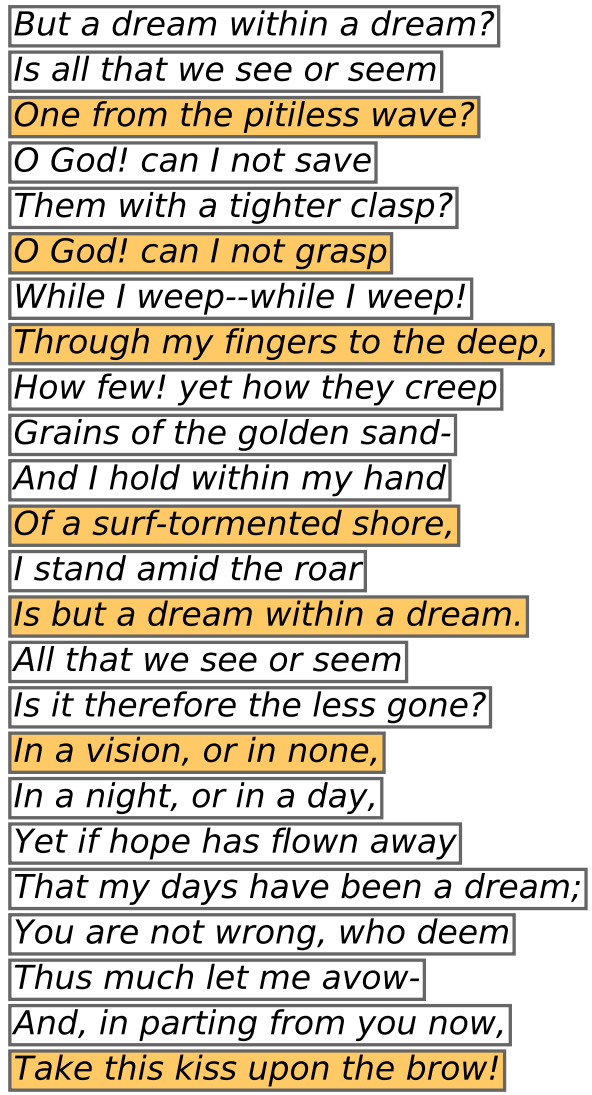


Figure 14: Iteration 6

zonoSampling in action

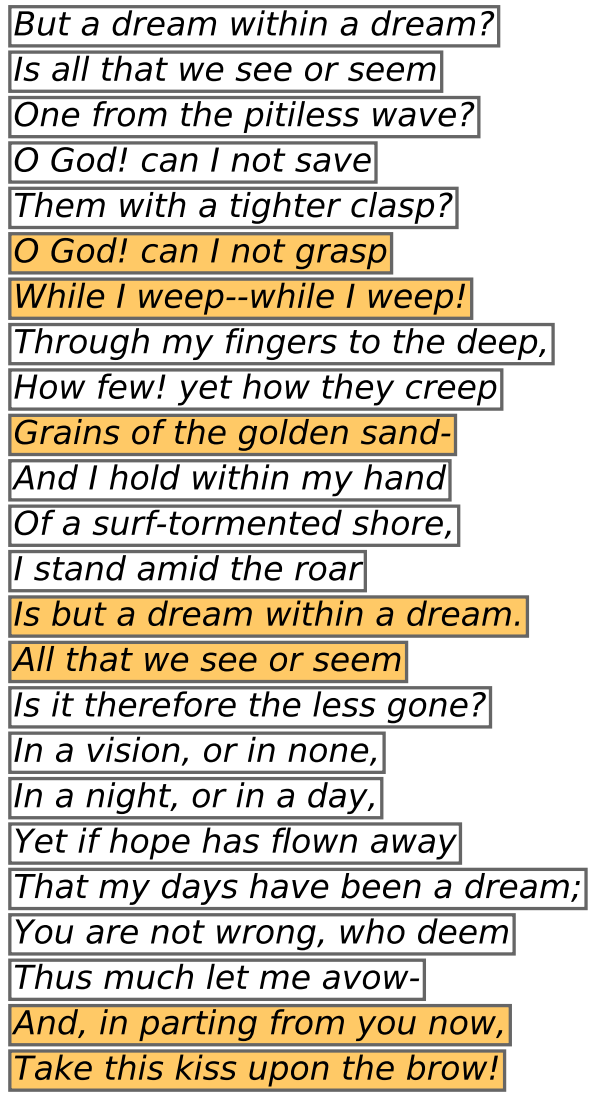


Figure 14: Iteration 7

zonoSampling in action

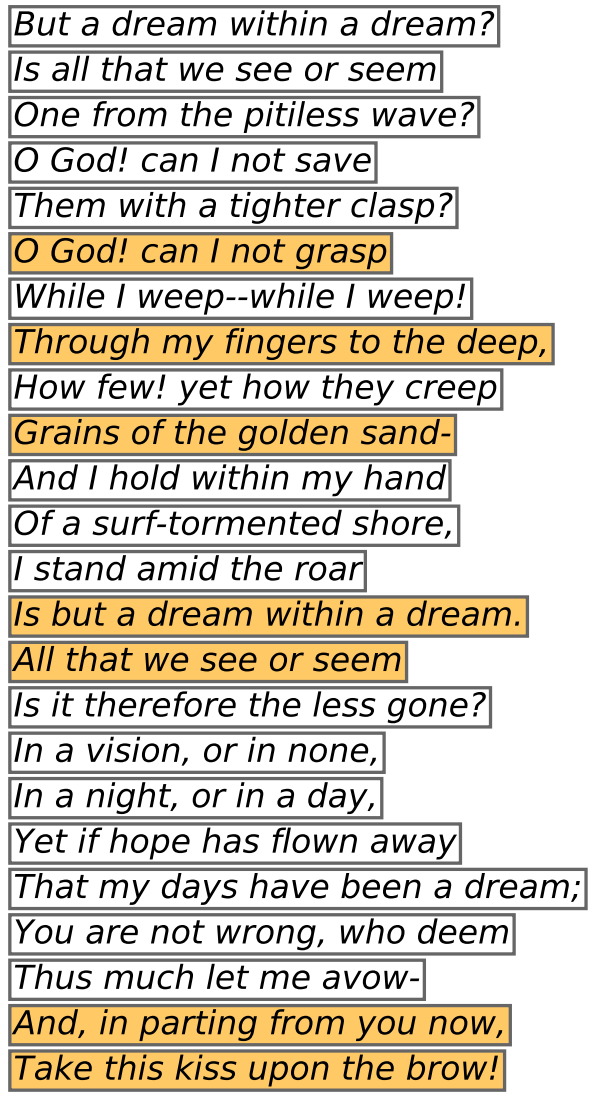


Figure 14: Iteration 8

zonoSampling in action

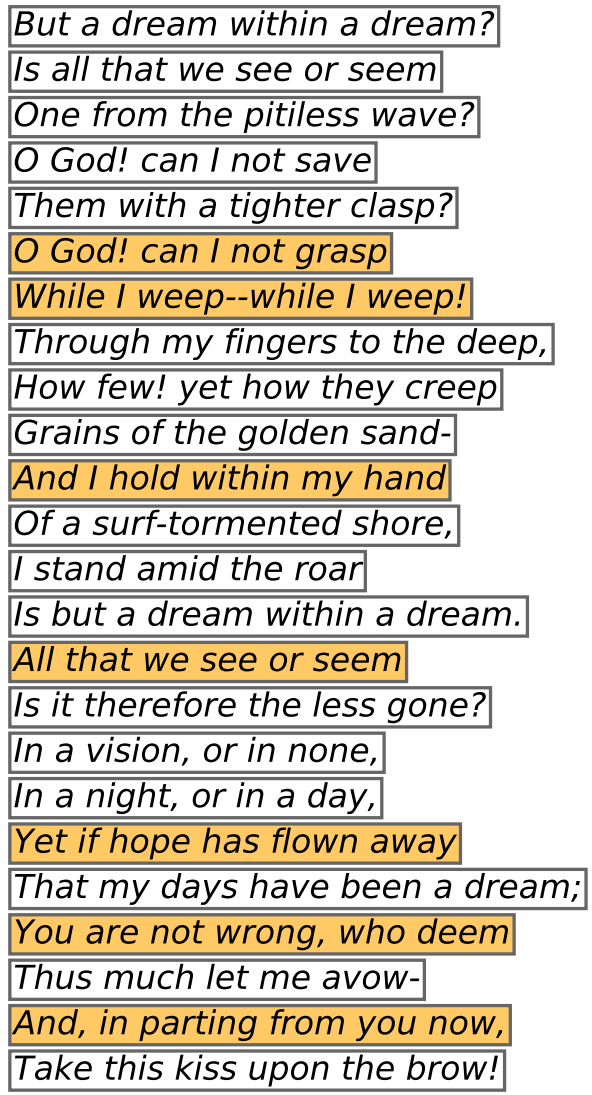


Figure 14: Iteration 9

zonoSampling in action

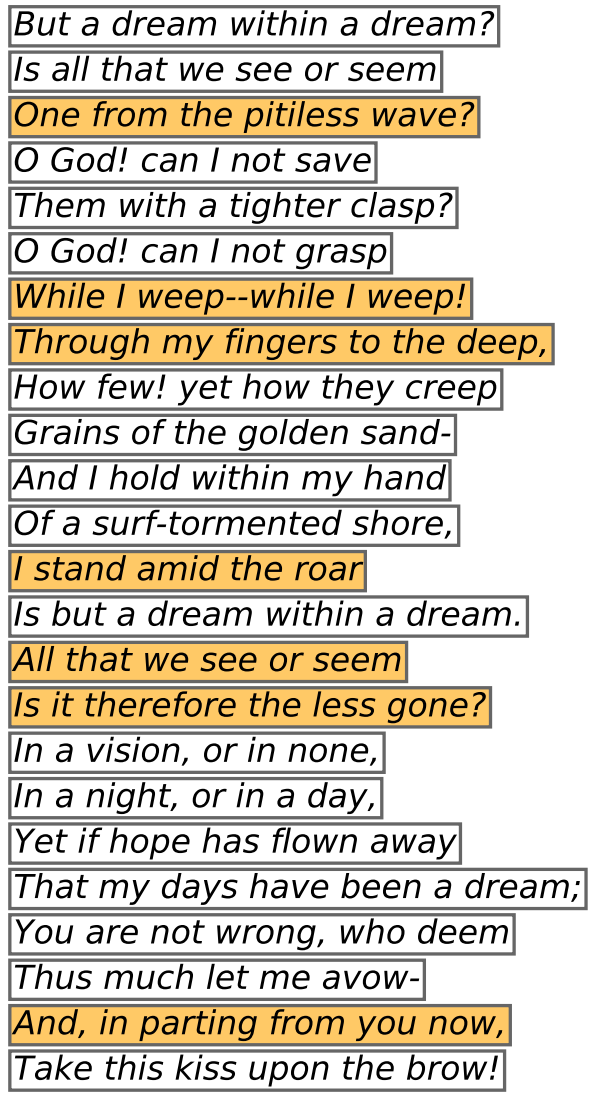


Figure 14: Iteration 10

Behaviour of our chain vs. iterations

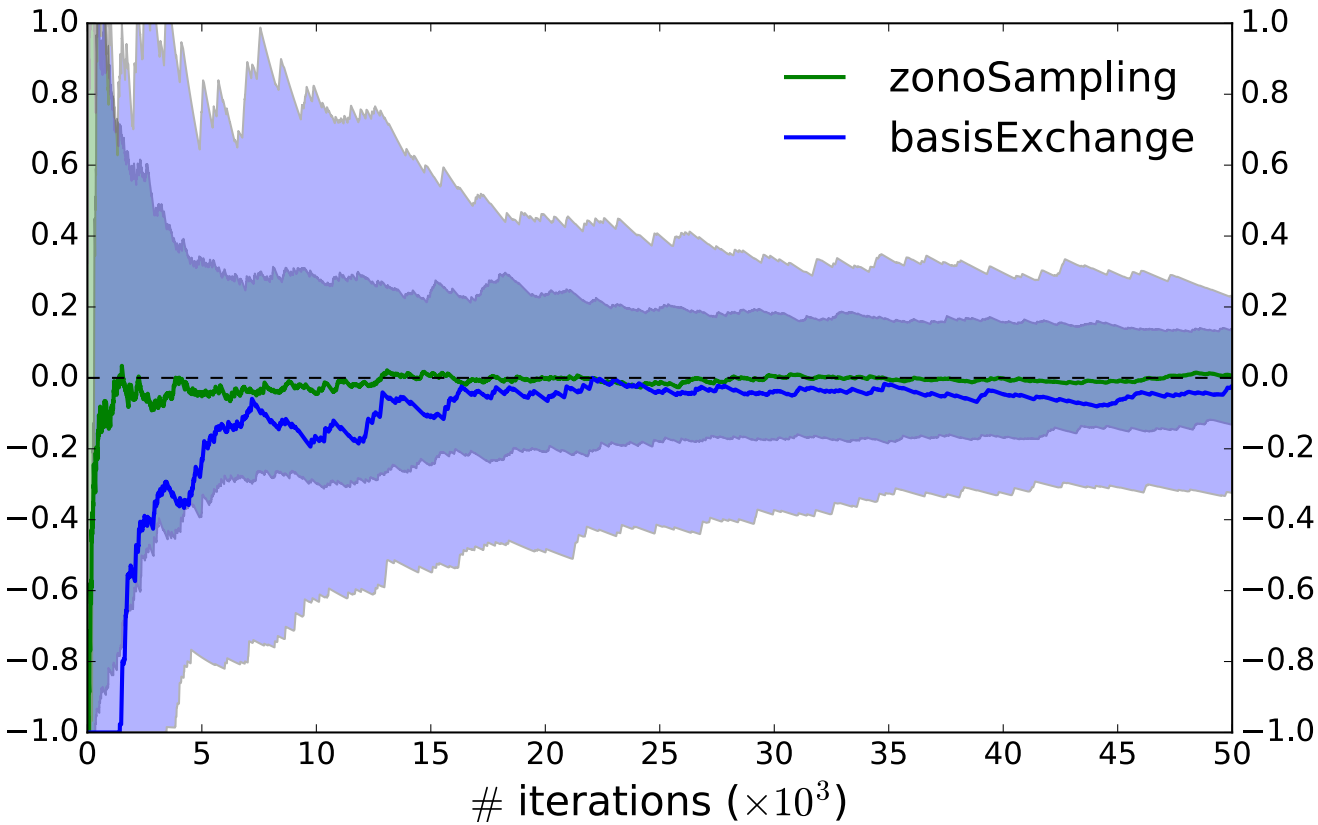


Figure 15: Relative error of the estimation $\mathbb{P}\left[\{i_1, i_2, i_3\} \subseteq \mathcal{X}\right]$ vs. iterations

Behaviour of our chain vs. CPU time

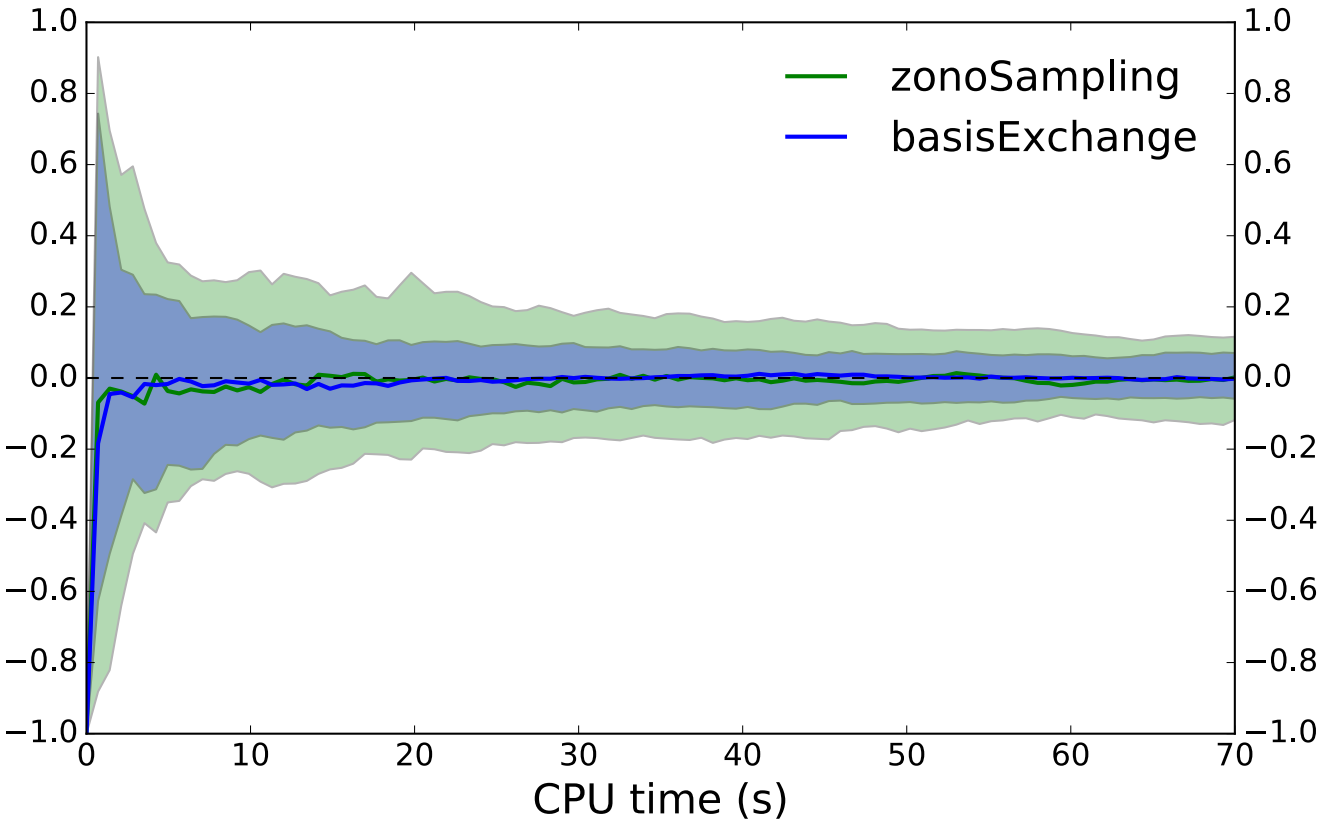


Figure 16: Relative error of the estimation $\mathbb{P}\left[\{i_1, i_2, i_3\} \subseteq \mathcal{X}\right]$ vs. CPU time

Summarizing a news article from Slate

Find Y to maximize (Kulesza and Taskar, 2012)

$$\int \textsc{Rouge-1F}(Y, Z)\,\mathrm{DPP}(Z)dZ \approx \frac{1}{N}\sum_{i=1}^{N} \textsc{Rouge-1F}\,(Y, Y_i)$$

where Y_i are samples from our hit-and-run Markov chain

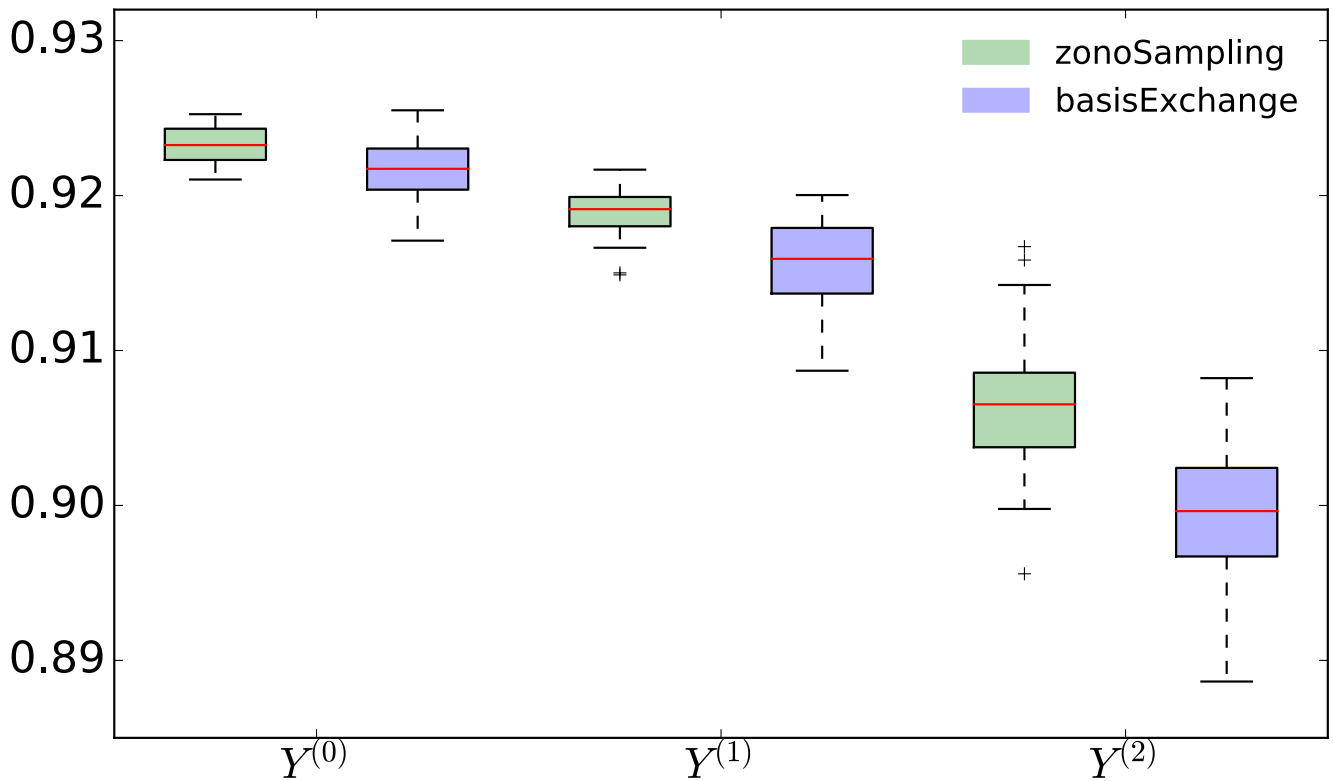


Figure 17: Estimation of the integrated cost

Conclusion

- Combinatorial geometry yields fast sampling of projection DPPs, provided we know A such that $\mathbf{K} = \mathbf{A}^T(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}$.
- Applications in ML (Kathuria et al., 2016; Kulesza and Taskar, 2012), graph sampling (Tremblay et al., 2017), Monte Carlo with DPPs (Bardenet and Hardy, 2016)...
- k-DPPs (Kulesza and Taskar, 2011)
- Potential speed-up?
  - Starting points
  - Specific implementation of the simplex method
- Mixing time?
  - Difficult: hit-and-run + LPs
  - Choice of linear objective c in LP (identification of the tile)
- All details in (Gautier et al., 2017)

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Continuous case

Spectrum of random matrices

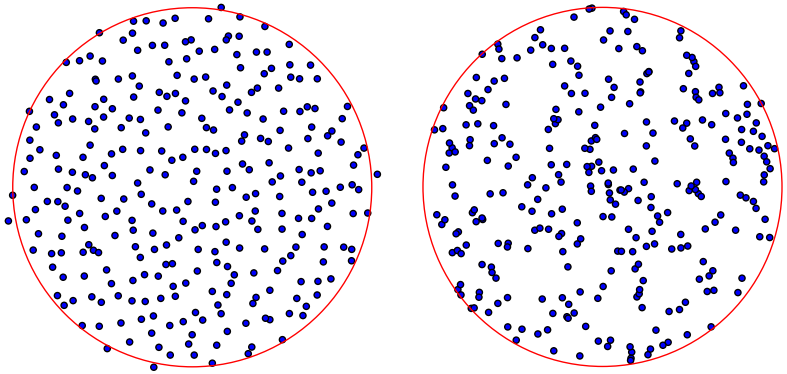

Figure 18: GUE vs. uniform

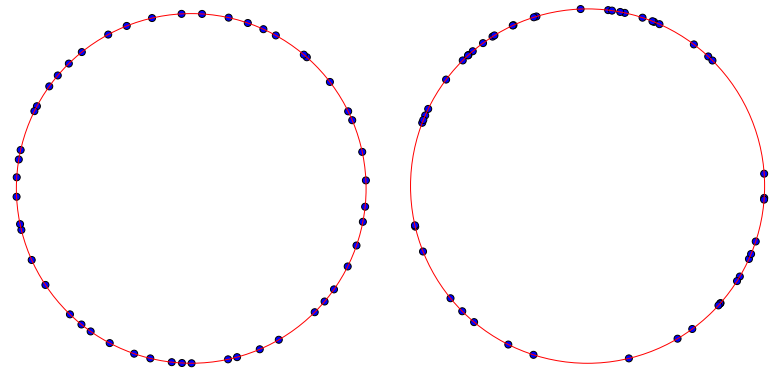

Figure 19: CUE vs. uniform

Non-intersecting paths

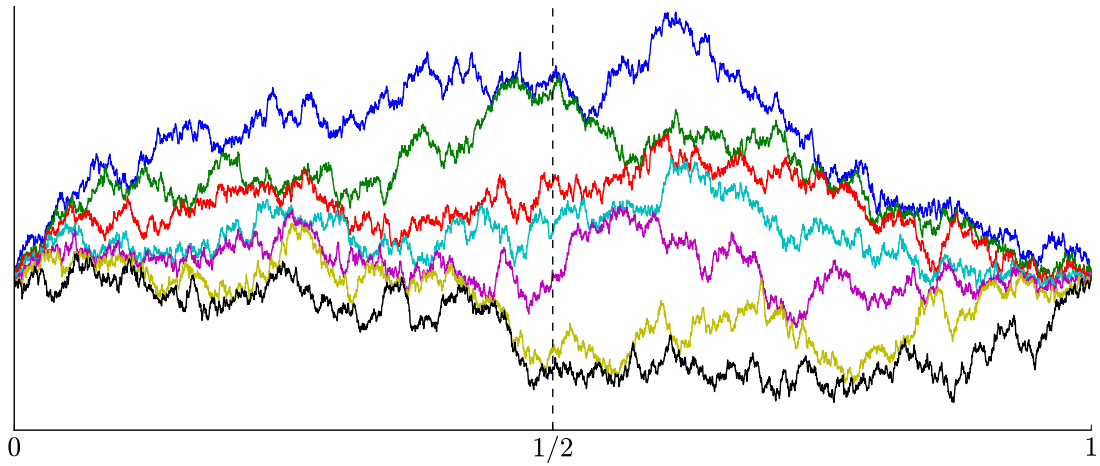

Figure 20: Dyson's Brownian motion

Summarizing a news article from Slate

If you are a scientist, this disregard for evidence probably drives you crazy.

So what do you do about it?

Across the country, science communication and advocacy groups report upticks in interest.

In 2010, Dan Kahan, a Yale psychologist, essentially proved this theory wrong.

If the deficit model were correct, Kahan reasoned, then people with increased scientific literacy, regardless of worldview, should agree with scientists that climate change poses a serious risk to humanity.

Scientific literacy, it seemed, increased polarization.

This lumps scientists in with the nebulous "left" and, as Daniel Engber pointed out here in Slate about the upcoming March for Science, rebrands scientific authority as just another form of elitism.

Is it any surprise, then, that lectures from scientists built on the premise that they simply know more (even if it's true) fail to convince this audience?

With that in mind, it may be more worthwhile to figure out how to talk about science with people they already know, through, say, local and community interactions, than it is to try to publish explainers on national news sites.

Goldman also said scientists can do more than just educate the public: The Union of Concerned Scientists, for example, has created a science watchdogteam that keeps tabs on the activities of federal agencies.

There's also a certain irony that, right here in this article, I'm lecturing scientists about what they might not know-in other words, I'm guilty of following the deficit model myself.

Figure 21: Using zonotope hit-and-run

Summarizing a news article from Slate

If you consider yourself to have even a passing familiarity with science, you likely find yourself in a state of disbelief as the president of the United States calls climate scientists "hoaxsters" and pushes conspiracy theories about vaccines.

In fact, it's so wrong that it may have the opposite effect of what they're trying to achieve.

Respondents who knew more about science generally, regardless of political leaning, were better able to identify the scientific consensus-in other words, the polarization disappeared.

In fact, well-meaning attempts by scientists to inform the public might even backfire.

Psychologists, aptly, dubbed this the "backfire effect."

But if scientists are motivated to change minds-and many enrolled in science communication workshops do seem to have this goal-they will be sorely disappointed.

That's not to say scientists should return to the bench and keep their mouths shut.

Goldman also said scientists can do more than just educate the public: The Union of Concerned Scientists, for example, has created a science watchdogteam that keeps tabs on the activities of federal agencies.

But I'm learning to better challenge scientists' assumptions about how communication works.

It's very logical, and my hunch is that it comes naturally to scientists because most have largely spent their lives in school-whether as students, professors, or mentors-and the deficit model perfectly explains how a scientist learns science.

So in the spirit of doing better, I'll not just write this article but also take the time to talk to scientists in person about how to communicate science strategically and to explain why it matters.

And it's not just Trump-plenty of people across the political spectrum hold bizarre and inaccurate ideas about science, from climate change and vaccines to guns and genetically modified organisms.

It seems many scientists would take matters into their own hands by learning how to better communicate their subject to the masses.

I've always had a handful of intrepid graduate students, but now, fueled by the Trump administration's Etch A Sketch relationship to facts, record numbers of scientists are setting aside the pipette for the pen.

This is because the way most scientists think about science communication-that just explaining the real science better will help-is plain wrong.

Before getting fired up to set the scientific record straight, scientists would do well to first considerthe science of science communication.

If the deficit model were correct, Kahan reasoned, then people with increased scientific literacy, regardless of worldview, should agree with scientists that climate change poses a serious risk to humanity.

Scientific literacy, it seemed, increased polarization.

Presenting facts that conflict with an individual's worldview, it turns out, can cause people to dig in further.

I spoke with Gretchen Goldman, research director of the Union of Concerned Scientists' Center for Science and Democracy, which offers communication and advocacy workshops.

Communication that appeals to values, not just intellect, research shows, can be far more effective.

I hope they end up doing the same.