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Lie Algebras Generated by Extremal Elements

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We study Lie algebras generated by extremal elements (i.e., elements spanning inner ideals) over a field of characteristic distinct from 2. There is an associative bilinear form on such a Lie algebra; we study its connections with the Killing form. Any Lie algebra generated by a finite number of extremal elements is finite dimensional. The minimal numbers of extremal generators for the Lie algebras of type A_n $(n \geq 1)$, B_n $(n \geq 3)$, C_n $(n \geq 2)$, D_n $(n \geq 4)$, E_n $(n = 6, 7, 8)$, F_4 and G_2 are shown to be $n+1$, $n+1$, $2n$, n, 5, 5, and 4 in the respective cases. These results are related to group theoretic ones for the corresponding Chevalley groups. © 2001 Academic Press

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1. INTRODUCTION

Let k be a field of characteristic not 2 and L a Lie algebra over k. We study the role of *extremal* elements in L, that is, those $x \in L$ with $[x, [x, L]] \subseteq kx$. Since an inner ideal of L is by definition [1] a linear subspace I of L such that $[I, [I, L]] \subseteq I$, this amounts to kx being an inner ideal. By $\mathscr{E}$ or, if necessary to express dependence on L, by $\mathscr{E}(L)$ we shall denote the set of all nonzero extremal elements of L.

We are mostly interested here in Lie algebras generated by extremal elements. The main motivation stems from the fact that long root elements are extremal in Lie algebras of Chevalley type (i.e., those Lie algebras over k that are given by the multiplication table of a Chevalley basis, coming from a simple Lie algebra in characteristic 0). They were used by Chernousov [4] in his proof of the Hasse principle for E_8. The associated root groups were studied in a more abstract group theoretic setting in [15]. Sandwiches, that is, elements $x \in L$ with $[x, [x, L]] = 0$, are extremal elements of a special kind; they are prominent in the classification of finite dimensional simple modular Lie algebras over algebraically closed fields of characteristics 5 and 7 and could well be useful in a similar way for other positive characteristics; cf. [11]. Extremal elements themselves are common in Lie algebras, as, by [10], any finite-dimensional Lie algebra over an algebraically closed field of characteristic $p > 5$ has extremal elements.

We shall be particularly concerned with Lie algebras which are generated by a finite set of extremal elements. Such a Lie algebra has finite dimension; see Section 4. By the work of Zel'manov and Kostrikin [18], there is a universal Lie algebra $\mathscr{L}_r$ generated by a finite number of sandwich elements $x_1, \ldots, x_r \in \mathscr{L}_r$; it is nilpotent and of finite dimension. This leads to the question of a description of these universal Lie algebras. Some information about these for small values of r is given.

Using functions arising from the definition of extremal elements, we are able to define an associative bilinear form f on L. In particular, if $x \in \mathscr{E}$ and $y \in L$, then $[x, [x, y]]$ is a multiple of x and so we can set $[x, [x, y]] = f(x, y)x$. In Section 9 we discuss the connection of f with the Killing form and give some information about the radical of this form. In Section 8 we determine the minimal number of generators by extremal elements for the Lie algebras of Chevalley type. Exponentiation by ad_x for x an extremal element gives an automorphism of L. The group generated by these automorphisms satisfies some conditions defined by Timmesfeld [15], which we discuss in Section 10.

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2. GENERAL PROPERTIES OF EXTREMAL ELEMENTS

Throughout the remainder of this paper k is a field of characteristic distinct from 2 and L is a Lie algebra over k. By the linearity of $[x,[x,y]]$ in y an element x of L is extremal if and only if there is a linear functional $f_x : y \mapsto f_x(y)$ on L such that, for all $y \in L$, we have

$$[x,[x,y]] = f_x(y)x. \tag{1}$$

Note that $f_x(y) = 0$ if x and y commute.

The following three lemmas are known; see [4].

LEMMA 2.1. *If $x, y \in \mathcal{E}$ then $f_x(y) = f_y(x)$.*

LEMMA 2.2. *Let $x \in \mathcal{E}$. Then, for all $y, z \in L$,*

$$2[[x,y],[x,z]] = f_x([y,z])x + f_x(z)[x,y] - f_x(y)[x,z], \tag{2}$$

$$2[x,[y,[x,z]]] = f_x([y,z])x - f_x(z)[x,y] - f_x(y)[x,z]. \tag{3}$$

LEMMA 2.3. *Let $x, y \in \mathcal{E}$ and $z \in L$. Then*

$$2[[x,y],[x,[y,z]]] = f_y(z)f_x(y)x + f_x([y,z])[x,y] - f_x(y)[x,[y,z]], \tag{4}$$

$$2[[x,y],[[x,y],z]] = (f_x([y,z]) - f_y([x,z]))[x,y] + f_x(y)(f_x(z)y + f_y(z)x - [y,[x,z]] - [x,[y,z]]). \tag{5}$$

Proof. Expression (4) is the first identity of Lemma 2.2 with $[y,z]$ replacing z. Identity (5) follows from two applications of (4) and Jacobi. ∎

The lemmas above also make it clear why we are assuming that the characteristic of k is not 2.

Let $x \in \mathcal{E}$. Because $\mathrm{ad}_x^3 = 0$, the exponential of the derivation $s\,\mathrm{ad}_x$ for $s \in k$ is given by

$$\exp(x,s) := 1 + s\,\mathrm{ad}_x + \frac{s^2}{2}\mathrm{ad}_x^2.$$

This is an automorphism of L in view of Identity (2) of Lemma 2.2. Denote by G (or $G(L)$ if necessary) the group of automorphisms of L generated by all $\exp(x,s)$ for $x \in \mathcal{E}$ and $s \in k$.

We use these automorphisms to show that if a Lie algebra is generated (as a Lie algebra) by extremal elements then it is also linearly spanned by extremal elements.

LEMMA 2.4. *If L is generated as a Lie algebra by extremal elements then it is linearly spanned by the set $\mathcal{E}$ of all extremal elements.*

Proof. We use induction on the length of $z \in L$ as a bracketing of elements from $\mathcal{E}$ (that is, obtained by forming successive brackets of elements from $\mathcal{E}$). It readily reduces to the case where this length is 2, the case of length 1 being $z \in \mathcal{E}$. Suppose $z = [x, y]$ with $x, y \in \mathcal{E}$. Now let $m = \exp(x, 1)y = y + [x, y] + \frac{1}{2}f_x(y)x$. As $\exp(x, 1)$ is an automorphism m is extremal, and so $[x, y]$ is in the linear span of the extremal elements x, m, y. ∎

As a consequence of Lemma 2.4, the maps f_x ($x \in \mathcal{E}$) defined in the beginning of this section give rise to a symmetric bilinear associative form. Since its proof is a routine use of the identities above, we omit it.

THEOREM 2.5. *Suppose that L is generated by $\mathcal{E}$. There is a unique bilinear symmetric form $f : L \times L \to k$ such that, for each $x \in \mathcal{E}$, the linear form f_x coincides with $y \mapsto f(x, y)$. This form is associative, in the sense that $f(x, [y, z]) = f([x, y], z)$ for all $x, y, z \in L$.*

The following consequence is direct from Identity (5) of Lemma 2.3 and the above theorem.

COROLLARY 2.6. *Let $x, y \in \mathcal{E}$ with $f_x(y) = 0$ but $[x, y] \neq 0$. Then $[x, y] \in \mathcal{E}$, with $f_{[x,y]}(z) = \frac{1}{2}(f_x([y, z]) - f_y([x, z]))$ for $z \in L$. If, moreover, $f_x = 0$ then $[x, y] \in \mathcal{E}$ with $f_{[x,y]} = 0$.*

3. SOME EXAMPLES

We first discuss Lie algebras generated by two extremal elements.

LEMMA 3.1. *Let L be generated by two extremal elements $x, y \in \mathcal{E}$. Then one of the following three assertions holds.*

(i) *$L = kx + ky$ is Abelian, $\mathcal{E} = L \setminus \{0\}$, and $f = 0$.*

(ii) *$L = kx + ky + kz$ with $z = [x, y] \neq 0$, its center $Z(L)$ and its commutator subalgebra $[L, L]$ coinciding with kz, and $L/Z(L)$ Abelian. Thus, L is a Heisenberg algebra. Moreover, $\mathcal{E} = L \setminus \{0\}$ and $f = 0$.*

(iii) *$L \cong \mathfrak{sl}_2$ and $\mathcal{E}$ consists of all nilpotent elements of L. Thus,*

$$\mathcal{E} \cup \{0\} = kx \cup ky \cup \bigcup_{\delta \in k\setminus\{0\}} k\left(\delta x + \delta^{-1}\lambda y + [x, y]\right),$$

where $\lambda = \frac{1}{2}f(x, y) \neq 0$.

Proof. Clearly, x, y, and $[x, y]$ linearly span L. The rest is straightforward. ∎

EXAMPLE 3.2. To see that there are real differences in the characteristic 2 case, let us take $k = \mathbb{Z}/2\mathbb{Z}$ and look at the case where L is generated by two extremal elements x, y with $[x, [x, y]] = x$. Then $L = kx + k[x, y] + ky$ is not isomorphic to $\mathfrak{sl}_2$, as it is simple whereas $\mathfrak{sl}_2$ is not.

Next we deal with finite dimensional Lie algebras of Chevalley type. A long root element of such a Lie algebra is an element of the form x_α as explained below with α a long root or the image of such an element under a Lie algebra automorphism.

PROPOSITION 3.3. *Let L be a Lie algebra of Chevalley type over k. Then $\mathcal{E}$ contains the long root elements of L. In particular, L is generated by $\mathcal{E}$. Moreover, if k has characteristic distinct from* 2 *and* 3, *every element of $\mathcal{E}$ is a long root element.*

Proof. Consider a Chevalley basis B of L with respect to a given Cartan subalgebra H of L. Write Φ for the corresponding root system, and x_β for the element of B associated with a given root $\beta \in \Phi$. Suppose that α is a long root in Φ. Then, for $h \in H$, we have $[x_\alpha, [x_\alpha, h]] = \alpha(h)[x_\alpha, x_\alpha] = 0$. Moreover, for $\beta \in \Phi$, the sum $\beta + 2\alpha$ is not a root in Φ unless $\beta = -\alpha$, so $[x_\alpha, [x_\alpha, x_\beta]] = 0$ unless $\beta = -\alpha$, in which case the result is a multiple of x_α.

To see that L is generated by the long root elements, observe that L is generated by the root elements of the basis B and (by analysis of the Lie rank 2 case) that any short root element can be written as a sum of three long root elements, as is clear from the following computations in the Lie algebras of type B_2 and G_2, respectively.

For B_2,

$$\underbrace{\exp(x_{\varepsilon_1}, 1)x_{-\varepsilon_1+\varepsilon_2}}_{\text{long}} = \underbrace{x_{-\varepsilon_1+\varepsilon_2}}_{\text{long}} \pm \underbrace{x_{\varepsilon_2}}_{\text{short}} \pm \underbrace{x_{\varepsilon_1+\varepsilon_2}}_{\text{long}}.$$

For G_2,

$$\underbrace{\exp(x_\alpha, 1)x_\beta}_{\text{long}} + \underbrace{\exp(x_\alpha, -1)x_\beta}_{\text{long}} = 2 \underbrace{x_\beta}_{\text{long}} \pm 2 \underbrace{x_{2\alpha+\beta}}_{\text{short}}.$$

As for the last assertion, let $e \in \mathcal{E}$. Under the given characteristic restriction for k, the proof of Lemma 2.1 of [1] and the paragraph following Step 1 in the proof of Theorem 3.2 of [1] show that there exists a Cartan subalgebra of L with eigenspace ke. Thus e is a root element. If the diagram of L is not simply laced, short root elements of L are easily seen not to be extremal. Hence e is a long root element. ∎

We expect that, in characteristic 3, each extremal element is also a long root element. Such an assertion might follow from a classification of nilpotent elements in the Lie algebras of Chevalley type, but we have not been able to verify this completely.

4. L IS FINITE DIMENSIONAL IN THE FINITELY GENERATED CASE

The following theorem is immediate from the work of Zel'manov and Kostrikin [18].

THEOREM 4.1. *If L is generated as a Lie algebra by a finite number of extremal elements, then L is finite dimensional.*

Proof. Suppose that L is generated by extremal elements $x_1, \ldots, x_r$. Denote by f the associated bilinear form. We first consider the case in which f is identically zero.

LEMMA 4.2. *Suppose that L is generated by elements $x_1, \ldots, x_r$ with $\mathrm{ad}_{x_i}^2 = 0$ for $i = 1, \ldots, r$. Then L is finite dimensional and is nilpotent.*

Proof. This is Theorem 1 of [17] except in characteristic 3, which is covered by Theorem 1 of [18]. ∎

Let $\mathcal{L}_r$ be the universal Lie algebra generated by r elements $x_1, \ldots, x_r$ satisfying $\mathrm{ad}_{x_i}^2 = 0$ for $i = 1, \ldots, r$. In particular, $\mathcal{L}_r$ is the quotient of the free Lie algebra F generated by r elements $f_1, f_2, \ldots, f_r$ with respect to the ideal J generated by all $[f_i, [f_i, u]]$ where u is a bracketing in the f_i.

LEMMA 4.3. *Suppose that L is generated by r extremal elements where the values of f need not all be* 0. *Then the dimension of L is at most* $\dim \mathcal{L}_r$. *In particular, L is finite dimensional.*

Proof. For a nonzero bracketing in F, its length is understood to be the total number of the f_i which appear, counting multiplicities (e.g., the length of $[[f_1, f_2], [f_1, f_3]]$ is 4). Choose bracketings $w_1, \ldots, w_t$ in the f_i such that $\{w_1 + J, \ldots, w_t + J\}$ is a basis of F/J, and let U be the linear span of $w_1, \ldots, w_t$. Then $F = U + J$. Consider the surjective homomorphism $\tilde{}: F \to L$ determined by $f_i \mapsto x_i$. We claim that $\{\tilde{w}_1, \ldots, \tilde{w}_t\}$ linearly spans L; i.e., $\widetilde{U} = L$.

Any element of J is a linear combination of terms

$$[u_s, [\ldots, [u_1, [f_i, [f_i, u]]] \ldots]] \tag{6}$$

with $u, u_1, \ldots, u_s$ bracketings in the f_i. Suppose that $\widetilde{U}$ is properly contained in L. Then there exists a nonzero bracketing $g \in F$ of minimal length with $\widetilde{g} \notin \widetilde{U}$. We may write g as a linear combination of the w_i and terms T_i as in (6) with all w_i, T_i of the same length as g. Among these terms T_i there is a term $T = [u_s, [\ldots, [u_1, [f_i, [f_i, u]]] \ldots]]$ as in (6) with $\widetilde{T} \notin \widetilde{U}$. Set $T_0 := [u_s, [\ldots, [u_1, f_i] \ldots]]$. Then $f(x_i, \widetilde{u})\widetilde{T_0} = \widetilde{T} \notin \widetilde{U}$. Hence $f(x_i, \widetilde{u}) \neq 0$ and $\widetilde{T_0} \notin \widetilde{U}$ with T_0 a bracketing of shorter length than g, a contradiction with the choice of g. ∎

Combining these two lemmas proves Theorem 4.1. ∎

Remark 4.4. We have worked out the Lie algebra for up to five generators and summarize the dimensions below. Let $\mathcal{L}_r$ be the universal Lie algebra generated by r extremal generators subject to f being identically 0. Also, let $\mathcal{R}_r$ be the free associative algebra over k generated by r elements $y_1, y_2, \ldots, y_r$ for which $y_i^2 = 0$ and $y_i w y_i = 0$ where w is any bracketing in the y_is. Then $\mathcal{L}_r/Z(\mathcal{L}_r)$, the image of $\mathcal{L}_r$ under ad, is a quotient of $\mathcal{R}_r$ in view of (3) of Lemma 2.2. Our computations have found the values of the corresponding dimensions in the table below.

Number of Generators	$\dim \mathcal{L}_r$	$\dim \mathcal{R}_r$
1	1	2
2	3	5
3	8	19
4	28	193
5	537	?

The work of [16] gives upper bounds for $\dim \mathcal{L}_r$ which are much higher than these actual values.

By way of example, consider the case $r = 2$. Put $A_1 = \mathrm{ad}_{x_1}$ and $A_2 = \mathrm{ad}_{x_2}$. Then a basis for $\mathcal{L}_2$ is $x_1, x_2, [x_1, x_2]$ of size 3. A basis for $\mathcal{R}_2$ is I, A_1, A_2, A_1A_2, A_2A_1, of size 5, as can be seen immediately. This explains the corresponding entries for $r = 2$ in the above table.

We will explain in later sections how some of the other entries have been determined. For the remainder we refer the reader to the algorithmic methods described in [13] and [8] using Lyndon words and to the computational algebra packages GAP [5] (including the FPSLA program by Gerdt and Kornyak) and LiE [9].

Let Rad(f) denote the radical of f, that is, the set of all $y \in L$ for which $f(y, z) = 0$ for all $z \in L$.

COROLLARY 4.5. *If L is simple and generated by a finite number of extremal elements, then f is nondegenerate and there is no $x \in L$, $x \neq 0$, with $\mathrm{ad}_x^2 = 0$. (This means that L is nondegenerate in the sense of* [1].)

Proof. We know that f is nontrivial by Lemma 4.2. (Otherwise f is identically 0 and L is nilpotent.) Thus, Rad(f) is a proper ideal of L and so, as L is simple, Rad(f) = 0. This means that f is nondegenerate.

Suppose $x \in L$ with $\mathrm{ad}_x^2 = 0$. Then x is extremal and $[x, [x, y]] = 0$ for all $y \in L$. Hence $x \in \mathrm{Rad}(f)$. This means $x = 0$. ∎

Remark 4.6. By the above corollary and the proof of Theorem 3.2 in [1], for $p > 5$, the only modular finite dimensional simple Lie algebras over an algebraically closed field of characteristic p and generated by extremal elements are quotients of those of Chevalley type.

5. THE THREE GENERATOR CASE

Suppose that L is generated by three extremal elements x, y, z. We want to determine the various possibilities for L. The identities of Lemmas 2.2 and 2.3, together with the identity

$$\begin{aligned}2[[x,[y,z]],[y,[x,z]]] = -\frac{1}{2}\big(&f_y(z)f_x([y,z])x + f_x([y,z])f_x(z)y\\ &+ f_x([y,z])f_x(y)z) - f_y(z)f_x(z)[x,y]\\ &+ f_y(z)f_x(y)[x,z] - f_x(z)f_x(y)[y,z],\end{aligned}$$

show that

$$x, y, z, [x,y], [x,z], [y,z], [x,[y,z]], [y,[x,z]]$$

linearly span L. In particular, L is at most eight-dimensional. Hence, $\mathcal{L}_3$ is also of dimension at most 8. It is readily checked, though, that the above eight bracketings provide a basis in the free case, so $\dim \mathcal{L}_3 = 8$.

EXAMPLE 5.1. The algebra $\mathfrak{sl}_3$ can be generated by three elements. It can be realized as

$$x = \begin{pmatrix} 0 & 1 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, \quad y = \begin{pmatrix} 0 & 0 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, \quad z = \begin{pmatrix} 1 & 1 & 1\\ 1 & 1 & 1\\ -2 & -2 & -2\end{pmatrix}$$

In this example, we have

$$f(x,y) = f(x,z) = f(y,z) = -2 \quad \text{and} \quad f(x,[y,z]) = 0.$$

The actions of ad_x and ad_y on the linear generators of L can be fully described by means of the identities in terms of the four parameters $f(x,y)$, $f(x,z)$, $f(y,z)$, and $f(x,[y,z])$.

We can describe the four parameters pictorially by drawing a triangle with vertices x, y, and z and labeling the edge $\{x,y\}$ with the edge parameter $f(x,y)$, and so on, and putting the central parameter $f(x,[y,z])$ in the middle, with an indication of orientation (note that $f(x,[y,z]) = f([x,y],z) = f(z,[x,y])$, so the value is invariant under cyclic permutations of the nodes).

We shall reduce $f(x, [y, z])$ to zero by transforming the generators using elementary transformations. To begin reduction, consider the triple $x, y, \exp(x, s)z$, where $s \in k$. It has the parameters

$$\begin{aligned} f(x, y) &= f(x, y) \\ f(x, \exp(x, s)z) &= f(x, z) \\ f(y, \exp(x, s)z) &= f(y, z) - sf(x, [y, z]) + \frac{1}{2}s^2 f(x, y)f(x, z) \\ f(x, [y, \exp(x, s)z]) &= f(x, [y, z]) - sf(x, z)f(x, y). \end{aligned}$$

Clearly, this triple again consists of extremal elements and generates the same algebra as x, y, and z. If at least two of the three edges have nonzero labels (e.g., $f(x, z)$ and $f(x, y)$), then we can transform the central parameter to 0 (by taking $s = f(x, [y, z])/(f(x, z)f(x, y))$).

On the other hand, if at most one edge is nonzero, say $f(x, y)$, and the central parameter $f(x, [y, z])$ is also nonzero, then the above transformation shows that we can move to three extremal generators x, y, $\exp(x, s)z$ with one more edge (namely, $f(y, \exp(x, s)z)$ and, if applicable, $f(x, y)$) nonzero. Hence, we can reduce to the previous case (if necessary in two steps), and so we may assume that the central parameter is zero: $f(x, [y, z]) = 0$.

Next, we scale x, y, z to αx, βy, γz for nonzero $\alpha, \beta, \gamma \in k$. This leaves $f(\alpha x, \beta\gamma[y, z]) = 0$ and changes the edge labels to

$$\alpha\beta f(x, y), \quad \alpha\gamma f(x, z), \quad \beta\gamma f(y, z).$$

We claim that, at the cost of a field extension of k, all nonzero edge labels may be transformed into -2. If at least one of them is zero, this is obvious. Otherwise, take

$$\alpha = \sqrt{\frac{-2f(y, z)}{f(x, y)f(x, z)}}, \quad \beta = \sqrt{\frac{-2f(x, z)}{f(x, y)f(y, z)}}, \quad \gamma = \sqrt{\frac{-2f(x, y)}{f(x, z)f(y, z)}}.$$

Thus we are left with four essentially different cases, distinguished by the number of nonzero labeled edges in the triangle. Straightforward computation using GAP [5] leads to the following descriptions of the resulting four Lie algebras.

THEOREM 5.2. *Suppose that L is generated by three extremal elements. Then after extending the field if necessary, L is generated by three extremal elements whose central parameter is zero and whose nonzero edge parameters are -2. In particular, L is a quotient of a Lie algebra M generated by extremal elements x, y, z with $f(x, [y, z]) = 0$ and* $\dim M = 8$. *Moreover, according to whether the number of nonzero edge parameters is* 0, 1, 2, *or* 3, *the Lie algebra M has the form* (0), (1), (2), *or* (3) *below.*

(0) $f = 0$ *and* $M \cong \mathscr{L}_3$. *Thus, M is nilpotent, with* $[M, M] = k[x, y] + k[x, z] + k[y, z] + Z$ *where* $Z = k[x, [y, z]] + k[y, [x, z]] = [[M, M], M]$ *is the center of* M.

(1) $f(x, y) = -2$, $f(x, z) = f(y, z) = 0$, *and* $M = Z \oplus [M, M]$ *where* $Z = k(z - [x, [y, z]] - [y, [x, z]])$ *is the center of M. The solvable radical of M is* $R = kz + k[x, z] + k[y, z] + k[x, [y, z]] + k[y, [x, z]]$. *The subalgebra* $S = kx + k[x, y] + ky$ *is isomorphic to* $\mathfrak{sl}_2$ *and M is the semi-direct product of S and R. The S-modules* $k[x, z] + k[y, [x, z]]$ *and* $k[y, z] + k[x, [y, z]]$ *are irreducible.*

(2) $f(x, y) = f(x, z) = -2$, $f(y, z) = 0$, *and M is the semi-direct product of* $S = kx + k[x, y] + ky$, *which is isomorphic to* $\mathfrak{sl}_2$, *and the solvable radical R of M. Moreover,* $R = k(y - \frac{1}{2}[y, [x, z]]) + k(z - \frac{1}{2}[y, [x, z]]) + k([x, y] - [x, z]) + k[y, z] + k[x, [y, z]]$, $[R, R] = k(y + z - [y, [x, z]]) + k[y, z] + k[x, [y, z]]$, *and* $[R, [R, R]] = k[y, z] + k[x, [y, z]]$. *The center of M is trivial and* $[M, M] = M$. *The subspace* $k(y + z - [x, [y, z]] - [y, [x, z]])$ *of* $[R, R]$ *is centralized by S.*

(3) $f(x, y) = f(x, z) = f(y, z) = -2$ *and* $M \cong \mathfrak{sl}_3$ *as described in Example* 5.1.

The algebra $\mathscr{R}_3$ can be determined easily. Again let $A_i = \mathrm{ad}_{x_i}$.

Words	Conditions	Number
I	identity	1
A_i		3
A_iA_j	i, j distinct	6
$A_iA_jA_k$	i, j, k distinct	6
$A_iA_jA_kA_i$	i, j, k distinct	3

Note that in the last line as $A_i[A_j, A_k]A_i = 0$ we get $A_iA_jA_kA_i = A_iA_kA_jA_i$ and so there are only three of these. Note that these are the only possibilities as multiplying one of the words of length 4 by any A_l obviously gives 0 on one side and gives 0 on the other side after using the commutation rule just given. There is such an algebra, as we found by using the four generator Lie algebra $\mathscr{L}_4$, to be described in the next section.

6. THE FOUR GENERATOR CASE

Definition 6.1. A *monomial of length s* is a bracketing of the form

$$[x_1, [x_2, \ldots [x_{s-1}, x_s] \ldots]].$$

A monomial is *reducible* if it is a linear combination of monomials of strictly smaller length.

LEMMA 6.2. *Let L be generated by a subset D. Then L is the linear span of all monomials in elements of D.*

Proof. The proof is by induction on the length of a bracketing (with respect to D) and Jacobi. ∎

PROPOSITION 6.3. *Let L be generated by the extremal elements x, y, z, u. Then L is linearly spanned by the following 28 monomials of length ≤ 5:*

$$x, y, z, u,$$
$$[x, y], [x, z], [x, u], [y, z], [y, u], [z, u],$$
$$[x, [y, z]], [x, [y, u]], [x, [z, u]], [y, [x, z]], [y, [x, u]], [y, [z, u]],$$
$$[z, [x, u]], [z, [y, u]],$$
$$[x, [y, [z, u]]], [x, [z, [y, u]]], [y, [x, [z, u]]], [y, [z, [x, u]]],$$
$$[z, [x, [y, u]]], [z, [y, [x, u]]],$$
$$[x, [y, [z, [x, u]]]], [y, [x, [z, [y, u]]]], [z, [x, [y, [z, u]]]],$$
$$[u, [x, [y, [z, u]]]].$$

Proof. Since L is linearly spanned by the monomials in x, y, z, u, we have to show that each monomial may be written as a linear combination of the given 28 elements.

All monomials of length 1 are on the list. There are $4 \cdot 3$ monomials of length 2 with different factors. Since $[a, b] = -[b, a]$ for all $a, b \in L$, all of them may be expressed by the six monomials of length 2 on the list.

All monomials of length 3 which involve only two letters are reducible, since x, y, z, u are extremal. There are $4 \cdot 3 \cdot 2$ monomials of length 3 with three different letters. With antisymmetry, all $[x, [a, b]]$ and all $[y, [a, b]]$ may be expressed. By Jacobi this holds also for the remaining ones.

All monomials of length 4 which involve only three letters are reducible; see the identity for $2[x, [y, [x, z]]]$ in Lemma 2.2. There are $4 \cdot 3 \cdot 2$ monomials of length 4 with different factors.

With Jacobi we have the following equations

$$[x, [y, [z, u]]] - [x, [z, [y, u]]] + [x, [u, [y, z]]] = 0, \tag{7}$$
$$[y, [x, [z, u]]] - [y, [z, [x, u]]] + [y, [u, [x, z]]] = 0, \tag{8}$$
$$[z, [x, [y, u]]] - [z, [y, [x, u]]] + [z, [u, [x, y]]] = 0. \tag{9}$$

The elements in the first two columns are on the list. Hence all monomials beginning with x, y, or z may be expressed as a linear combination of the given 28 basis elements. For the monomials beginning with u, we calculate

$$[u,[x,[y,z]]]-[x,[u,[y,z]]]+[[y,z],[u,x]]=0, \tag{10}$$

$$[u,[y,[x,z]]]-[y,[u,[x,z]]]+[[x,z],[u,y]]=0, \tag{11}$$

$$[u,[z,[x,y]]]-[z,[u,[x,y]]]+[[x,y],[u,z]]=0. \tag{12}$$

The products of the form $[[a,b],[c,d]]$ may be expressed as

$$[x,[y,[z,u]]]-[y,[x,[z,u]]]+[[z,u],[x,y]]=0, \tag{13}$$

$$[x,[z,[y,u]]]-[z,[x,[y,u]]]+[[y,u],[x,z]]=0, \tag{14}$$

$$[y,[z,[x,u]]]-[z,[y,[x,u]]]+[[x,u],[y,z]]=0. \tag{15}$$

Hence also the monomials of length 4 beginning with u may be expressed as a linear combination of the monomials on the list.

We obtain monomials of length 5 by multiplying a letter from the left with a monomial $[a,[b,[c,d]]]$ of length 4. This yields four possibilities: first, $[a,[a,[b,[c,d]]]]$, which obviously is reducible; second, $[b,[a,[b,[c,d]]]]$, which is of the form $[b,[a,[b,e]]]$ and hence is reducible; third, $[c,[a,[b,[c,d]]]]$. There is no obvious way to rewrite this. And fourth we have $[d,[a,[b,[c,d]]]]$, which yields the previous case by interchanging c and d. We are left with the monomials of the form $[c,[a,[b,[c,d]]]]$ or $[d,[a,[b,[c,d]]]]$, where $[a,[b,[c,d]]]$ is one of the monomials of length 4 on the list. This yields the following 12 monomials $m_{11}, m_{12}, \ldots, m_{34}$ (m_{ij} is in row i and column j):

$$[z,[x,[y,[z,u]]]], [y,[x,[z,[y,u]]]], [z,[y,[x,[z,u]]]],$$
$$[x,[y,[z,[x,u]]]], [y,[z,[x,[y,u]]]], [x,[z,[y,[x,u]]]],$$
$$[u,[x,[y,[z,u]]]], [u,[x,[z,[y,u]]]], [u,[y,[x,[z,u]]]],$$
$$[u,[y,[z,[x,u]]]], [u,[z,[x,[y,u]]]], [u,[z,[y,[x,u]]]].$$

Our intended basis vectors are $m_{11}, m_{12}, m_{14}, m_{23}$. We may express the other eight elements as follows:

m_{24}:	with (7),
m_{13}, m_{31}:	with (13) and (12),
m_{32}:	with (8),
m_{21}, m_{33}:	with (14) and (11),
m_{22}, m_{34}:	with (15) and (10).

Finally, we show that all monomials of length 6 are reducible. We multiply the monomials of length 5 on the list with a letter. Note that all these monomials of length 5 are of the form $\pm[c, [a, [b, [c, d]]]]$. Multiplication from the left with c or a yields reducible monomials. Next, we deal with $[b, [c, [a, [b, [c, d]]]]]$. With Jacobi we may pass from $[a, [b, [c, d]]]$ to $[b, [[c, d], a]]$ and $[[c, d], [a, b]]$. From $[[c, d], [a, b]]$, we pass to $[d, [c, [a, b]]]$ and $[c, [[a, b], d]]$. Now we multiply first with c, then with b from the left. This yields products of the form $[b, [c, [b, [[c, d], a]]]]$, $[b, [c, [d, [c, [a, b]]]]]$, and $[b, [c, [c, [[a, b], d]]]]$, which are all reducible. (Look for patterns of the type $[u, [v, [u, w]]]$.) The last monomial we have to reduce is of the form $[d, [c, [a, [b, [c, d]]]]]$. We pass from $[b, [c, d]]$ by Jacobi to $v_1 = [c, [d, b]]$ and $v_2 = [d, [c, b]]$. The products $[d, [c, [a, v_1]]]$ and $[d, [c, [a, v_2]]]$ are reducible. (In the second one, the two letters d are at distance 3.)

As a consequence all monomials of length ≥ 6 are reducible and may be written as a linear combination of the 28 vectors on the list. ∎

Remark 6.4. Removing all elements involving a letter u from the list for the 4-generator case yields a spanning set of size 8 for the 3-generator case. More generally, for $r \in \mathbb{N}$, $r > 1$, the Lie algebra $\mathcal{L}_{r-1}$ is a Lie subalgebra of $\mathcal{L}_r$, generated by the first $r - 1$ elements of a set $\{x_1, \dots, x_r\}$ of extremal generators for $\mathcal{L}_r$. The module generated by x_r under the ad action of $\mathcal{L}_{r-1}$ gives the algebra $\mathcal{R}_{r-1}$ defined in Remark 4.4. (This can be proved by observing that the defining relations for $\mathcal{L}_r$ are homogeneous with respect to the multidegree counting the number of each x_i, so that a nontrivial relation amongst bracketings in $\mathcal{R}_{r-1}x_r$ would lead to a nontrivial homogeneous relation of x_r-degree 1. Thus, no relation in $\mathcal{R}_{r-1}x_r$ involves relations coming from $\mathrm{ad}_{x_r}^2 = 0$, and so any linear relation amongst bracketings in $\mathcal{R}_{r-1}x_r$ is a consequence of the defining relations of $\mathcal{R}_{r-1}$.)

Counting the monomials with a single u in the list of Proposition 6.3, we find $\dim \mathcal{R}_3 = 19$, distributed according to length as follows, where length of course stands for the length in $\mathcal{R}_3$, which is one less than the length of these monomials of Proposition 6.3.

Length:	0	1	2	3	4
Number	1	3	6	6	3

The algebra $\mathcal{R}_4$ has dimension 193. This can be seen by routine enumeration which we do not include. However, we list the numbers of words of each length in the table below.

Length:	0	1	2	3	4	5	6	7	8	9
Number	1	4	12	24	36	40	36	24	12	4

Note that in all of these algebras $\mathcal{R}_2$, $\mathcal{R}_3$, $\mathcal{R}_4$ the number of words with a given length is symmetric about the highest number after deleting the one word of length 0 which is the identity. We wonder if this behavior persists for $\mathcal{R}_m$ with $m \geq 5$.

7. QUADRATIC MODULES

DEFINITION 7.1. Let L be a Lie algebra generated by extremal elements. We call an L module U quadratic if there is a generating set D of extremal elements of L such that $x \cdot (x \cdot U) = 0$ for all $x \in D$.

Remark 7.2. Suppose that L is generated by n extremal elements. Consider the subalgebra A generated by the set D of any $n-1$ extremal generators of L. Since A is an A-submodule of L, the quotient L/A is also an A-module. Clearly, $x \cdot (x \cdot L/A) = 0$ for all $x \in D$, and $D \subseteq \mathcal{E}(L) \cap A \subseteq \mathcal{E}(A)$. Therefore, L/A is a quadratic A module.

Because of this it is natural to study quadratic modules. We give the irreducible quadratic modules for Lie algebras of Chevalley type.

PROPOSITION 7.3. *Let L be a Lie algebra of Chevalley type and k a field of characteristic distinct from* 2 *and* 3. *The highest weights of its nontrivial quadratic highest weight modules of finite dimension are given in the table below.*

Type of L	Highest Weights
A_n	$\omega_1, \ldots, \omega_n$
B_n	$\omega_1,\ \omega_n$
C_n	$\omega_1, \ldots, \omega_n$
D_n	$\omega_1, \omega_{n-1}, \omega_n$
E_6	ω_1, ω_6
E_7	ω_7
F_4	ω_4
G_2	ω_1

Proof Sketch. (A strongly related result for groups can be found in [12].) Let V be a nontrivial quadratic highest weight module for L of finite dimension. By Proposition 3.3, there is a single G orbit of extremal elements in L. Moreover, all these (long root) elements have the same nilpotency index on V (because the highest weight representation is equivalent to a composition of itself with conjugation by an element of G). Consequently, each element $x \in \mathcal{E}(A)$ satisfies $x \cdot (x \cdot V) = 0$.

In particular, for every long root α, the corresponding Chevalley basis element x_α is nilpotent of index 2. It follows that if μ is a weight of V then $\mu - 2\alpha$ is not. Since the set of weights for V is a convex subset of the coset of the root lattice containing the highest weight, only small weights can occur. Thus, the result can be readily established by use of the LiE, program, cf. [9], and the following argument for the case of a single root length. ∎

LEMMA 7.4. *Suppose that L is a Lie algebra of Chevalley type with only one root length. If U is an irreducible quadratic finite dimensional L-module, then U is a minuscule weight representation.*

Proof. Let λ be the highest weight of U. Suppose μ is a weight for U. Then there is a path of weights from μ to λ such that for each adjacent pair (μ_1, μ_2) from the path the difference $\mu_2 - \mu_1$ is a positive root. Since there is only one root length, each of these differences is a root whose root element is extremal. Now take the fundamental $\mathfrak{sl}_2$-triplet containing this root element. Quadraticity means that the nontrivial irreducible subrepresentations in U of the corresponding subalgebra isomorphic to $\mathfrak{sl}_2$ all have dimension 2. But μ_1, μ_2 are in the same representation, so they belong to a two-dimensional module. But then they are conjugate by an element of the corresponding subgroup of type A_1. This establishes that U has a basis of weight spaces which are all conjugate, which can be taken as a definition of minuscule. ∎

Remark 7.5. For the Lie algebra of type E_8 there are no nontrivial finite dimensional quadratic modules. Thus, this Lie algebra can only occur as a direct component of a bigger Lie algebra generated by extremal elements.

8. THE MINIMAL NUMBER OF EXTREMAL GENERATORS FOR LIE ALGEBRAS OF CHEVALLEY TYPE

Let $\mathfrak{g}$ be a Lie algebra of Chevalley type over k (of characteristic distinct from 2). As we have seen in Proposition 3.3, the long root elements of $\mathfrak{g}$ are extremal elements and $\mathfrak{g}$ is generated by these extremal elements. Write $t(\mathfrak{g})$ for the minimal number of these extremal generators of $\mathfrak{g}$. In this section, we determine $t(g)$. We discuss implications for the group analog in Section 10.

Fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and let Φ be the corresponding root system. Throughout this section x_α and h_α denote, respectively, the root element and the element in $\mathfrak{h}$ of $\mathfrak{g}$ corresponding to the root α in Φ. Also, $\exp(x_\alpha)$ denotes $\exp(x_\alpha, 1)$ in the notation of Section 2. The signs appearing here in the multiplication rules for the x_α are the ones arising from the implementations in the relevant packages. In general, the signs depend on the choice of a Chevalley basis.

LEMMA 8.1. *Let* $\mathfrak{g}$ *be a Lie algebra of Chevalley type and let* $\alpha_1, \ldots, \alpha_n$ *be the simple roots in* Φ. *Denote by* β *the root of highest height. Then* $\mathfrak{g}$ *is generated by the root elements* $x_{\alpha_1}, x_{\alpha_2}, \ldots, x_{\alpha_n}, x_{-\beta}$.

Proof. The simple root elements with respect to Φ generate the subalgebra $\mathfrak{n} = \sum_{\alpha\in\Phi^+} \mathfrak{g}_\alpha$ where Φ^+ is the set of positive roots and $\mathfrak{g}_\alpha$ is a root space. The lemma follows from the fact that the $\mathfrak{n}$-submodule of $\mathfrak{g}$, generated by the root element corresponding to the root of lowest height, coincides with $\mathfrak{g}$. ∎

THEOREM 8.2. *Let* $\mathfrak{g}$ *be a Lie algebra of Chevalley type over the field* k *of characteristic distinct from* 2. *Then the number* $t(\mathfrak{g})$ *is as given in the table*

Type of $\mathfrak{g}$	$t(\mathfrak{g})$	*Condition*
A_n	$n+1$	$n \geq 1$
B_n	$n+1$	$n \geq 3$
C_n	$2n$	$n \geq 2$
D_n	n	$n \geq 4$
E_n	5	$n = 6, 7, 8$
F_4	5	
G_2	4	

The proof of the theorem will be given in the rest of this section.

LEMMA 8.3. *If* $\mathfrak{g}$ *has an irreducible representation of dimension* N *and its extremal elements have rank* m *in this representation, then the number of extremal elements generating* $\mathfrak{g}$ *is at least* N/m.

Proof. If $\mathfrak{g}$ is generated by the extremal elements $x_1, \ldots, x_t$, then the image of $\mathfrak{g}$ in V, the underlying N-dimensional vector space, is generated by

$$\langle \operatorname{Im} x_1, \ldots, \operatorname{Im} x_t\rangle = \langle \operatorname{Im} x_1\rangle + \cdots + \langle \operatorname{Im} x_t\rangle,$$

of dimension at most tm. Thus, irreducibility of the representation implies $tm \geq N$; whence we have the lemma. ∎

Here is an upper bound for algebras without multiple bonds.

LEMMA 8.4. *If the root system of* $\mathfrak{g}$ *has just one root length, then* $t(\mathfrak{g}) \leq n+1$, *where* n *is the rank of* $\mathfrak{g}$.

Proof. This follows immediately from Lemma 8.1. ∎

LEMMA 8.5. *If* $\mathfrak{g}$ *is generated by* t *extremal elements, then* $t(\mathfrak{g}) \geq 4$ *if* $\dim \mathfrak{g} \geq 9$, *and* $t(\mathfrak{g}) \geq 5$ *if* $\dim \mathfrak{g} \geq 29$.

Proof. This is immediate from the dimensions given in Section 5 and in Proposition 6.3. ∎

The previous lemmas suffice for the proof of all exact lower bounds on $t(\mathfrak{g})$.

For A_n, consider the natural representation of dimension $N = n+1$. Since extremal elements have rank 1 in this representation, Lemma 8.3 gives $t(A_n) \geq n+1$.

For B_n, Lemma 8.3 applied to the natural module for $\mathfrak{g}$ (of dimension $2n+1$, in which the extremal elements have rank 2), we find $t(\mathfrak{g}) \geq (2n+1)/2$; whence $t(B_n) \geq n+1$. Observe that for $n = 2$, this bound is not sharp. Since the Lie algebra of type B_2 has dimension 10, at least four extremal generators are needed in view of Lemma 8.5. This lower bound coincides with the result for the Lie algebra of type C_2 (which is isomorphic to the one of type B_2).

For C_n, Lemma 8.3 applied to the natural module for $\mathfrak{g}$ (of dimension $2n$, in which the extremal elements have rank 1), we find $t(C_n) \geq 2n$.

For D_n, recall that the Lie algebra has a natural representation of dimension $2n$, in which extremal elements have rank 2; hence, by Lemma 8.3, $t(D_n) \geq n$.

For the exceptional Lie algebras, Lemma 8.5 shows that the Lie algebra of type G_2 is generated by no fewer than four extremal elements and the other four (types F_4 and E_6, E_7, E_8) are generated by no fewer than five extremal elements. When k has characteristic 3, the Lie algebra $\mathfrak{g}$ of Chevalley type G_2 has a seven-dimensional simple quotient. This quotient is isomorphic to a quotient of $\mathfrak{sl}_3$ by its center, and is generated by three extremal elements. Nevertheless, $\mathfrak{g}$ itself, being 14-dimensional, cannot be generated by fewer than four extremal elements.

To prove the theorem, it remains to show that there is a generating set of extremal elements of the size indicated in the table. To this end, we often argue along the following lines. Let $\mathfrak{g}$ be the Lie algebra under consideration. We shall work with a fixed Chevalley basis $\mathcal{B}$ of $\mathfrak{g}$ and a fixed root system whose simple roots $\alpha_1, \ldots, \alpha_n$ are labeled as in [3]. When we talk about root elements corresponding to specific roots we mean elements from this basis.

We shall select a Lie subalgebra M of $\mathfrak{g}$ of Chevalley type generated by extremal elements, usually chosen from $\mathcal{B}$. Let $\mathfrak{g} = M \oplus V_1 \oplus \cdots \oplus V_r$ be the decomposition of the M-module $\mathfrak{g}$ into irreducible M-modules. The modules V_i are often spanned by certain elements of $\mathcal{B}$ and usually the index i corresponds to coefficients of simple roots not supported in M. We shall write down an element d which is the image of an extremal element under a composition of exponentials of root elements. It is an extremal element whose projections onto many of the modules V_j $(j = 1, \ldots, r)$ are nonzero. Next, we take C to be the Lie algebra generated by M and d, and, by suitably bracketing d by elements of C, we find vectors from each V_j, usually again elements of $\mathcal{B}$. Once those are found, we see that C

coincides with $\mathfrak{g}$, so $t(\mathfrak{g}) \leq t(M)+1$. (In the case $\mathfrak{g}$ is of type C_n, we need two additional extremal elements instead of one outside M, and there we show that $t(\mathfrak{g}) \leq t(M)+2$.)

We start with the classical Lie algebras. In this case, with the exception of A_n, we will write the simple roots using the unit orthonormal vectors in $\mathbb{C}^n$, $\varepsilon_1, \ldots, \varepsilon_n$ with α_i's expressed in terms of ε_i's as in Bourbaki. To treat the algebras F_4, E_6, E_7, and E_8 we use the fact that each contains a D_4 which is generated by four extremal elements. We then look for a fifth "generic" additional vector which has components in all factors of a decomposition of L/D_4 as a D_4 module.

All computations were made using the ELIAS routines [6] in GAP [5] and LiE [9]. The signs appearing here are the ones arising from the implementations in the relevant packages. In general, the signs depend on the choice of Chevalley basis.

8.1. *Type A_n.* By Lemma 8.4 we know that A_n can be generated by $n+1$ extremal elements, that is, $t(A_n) \leq n+1$. So there is nothing left to prove.

8.2. *Type B_n.* Let $\mathfrak{g}$ be the Lie algebra of type B_n.

First assume that $n=3$. Take M to be the Lie subalgebra of $\mathfrak{g}$ generated by the root elements corresponding to the first two simple roots, $\varepsilon_1-\varepsilon_2$ and $\varepsilon_2-\varepsilon_3$, and their negatives. Then M is of type A_2 and contains the root elements $x_{\pm(\varepsilon_1-\varepsilon_2)}$, $x_{\pm(\varepsilon_2-\varepsilon_3)}$, and $x_{\pm(\varepsilon_1-\varepsilon_3)}$. Moreover, $\mathfrak{g}$ decomposes as

$$\mathfrak{g} = M \oplus V_{-2} \oplus V_{-1} \oplus V_1 \oplus V_2 \oplus H_0,$$

where V_1 is the natural M-module linearly spanned by root elements x_{ε_1}, x_{ε_2}, and x_{ε_3}; V_2 is spanned by root elements $x_{\varepsilon_1+\varepsilon_2}$, $x_{\varepsilon_1+\varepsilon_3}$, and $x_{\varepsilon_2+\varepsilon_3}$. Their dual M-modules V_{-1} and V_{-2} are natural M-modules linearly spanned by $x_{-\varepsilon_1}$, $x_{-\varepsilon_2}$, $x_{-\varepsilon_3}$ and $x_{-(\varepsilon_1+\varepsilon_2)}$, $x_{-(\varepsilon_1+\varepsilon_3)}$, $x_{-(\varepsilon_2+\varepsilon_3)}$, respectively, and H_0 is the trivial M-submodule of L, spanned by the torus element h_{ε_3} centralizing M.

Now we write

$$\begin{aligned} d &= \exp(x_{-(\varepsilon_1+\varepsilon_2)})\exp(x_{\varepsilon_1})x_{-(\varepsilon_1-\varepsilon_2)} \\ &= x_{-(\varepsilon_1-\varepsilon_2)} - x_{\varepsilon_2} + x_{\varepsilon_1+\varepsilon_2} + x_{-\varepsilon_1} + h_{-(\varepsilon_1+\varepsilon_2)} - x_{-(\varepsilon_1+\varepsilon_2)}. \end{aligned}$$

Consider the Lie subalgebra C of $\mathfrak{g}$ generated by M and d. As $x_{\varepsilon_1-\varepsilon_2} \in M$, we have $-x_{\varepsilon_1} - x_{-\varepsilon_2} + M = [x_{\varepsilon_1-\varepsilon_2}, d] + M \subseteq C$, whence $x_{\varepsilon_1} + x_{-\varepsilon_2} \in C$. Bracketing by the element $x_{\varepsilon_2-\varepsilon_3}$ of M, we find $x_{-\varepsilon_3} \in C$. In particular, the M-submodule V_{-1} generated by this element belongs to C, whence also $x_{-\varepsilon_1}$, and so $e := d - x_{-(\varepsilon_1-\varepsilon_2)} - x_{-\varepsilon_1} \in C$.

Taking brackets of $x_{-\varepsilon_1}$ with e gives $2x_{-(\varepsilon_1-\varepsilon_2)} - x_{\varepsilon_2} + x_{-\varepsilon_1} \in C$, whence $x_{\varepsilon_2} \in C$. In particular, x_{ε_3}, which lies in the same M-submodule, belongs to C and hence so do all root elements corresponding to positive roots.

But then $e + C = x_{-(\varepsilon_1+\varepsilon_2)} + C \subseteq C$, and so also the lowest root element belongs to C. The root elements corresponding to the simple roots and to the lowest root generate $\mathfrak{g}$, so $C = \mathfrak{g}$. Thus $t(\mathfrak{g}) \leq t(M) + 1 = 4$.

Next, assume $n \geq 4$. In the Lie algebra $\mathfrak{g}$ of type B_n we have the Lie subalgebra M of type D_n generated by all long root elements (extremal elements) of the form $x_{\varepsilon_i \pm \varepsilon_j} (1 \leq i < j \leq n)$. As an M-representation space, $\mathfrak{g}$ decomposes into the adjoint module M and the natural representation V_1 of degree $2n$. Here, V_1 is linearly spanned by the elements $x_{\pm\varepsilon_j}$ ($j = 1, \ldots, n$) from the Chevalley basis $\mathscr{B}$.

Now

$$d := \exp(x_{\varepsilon_2}) x_{\varepsilon_1 - \varepsilon_2} = x_{\varepsilon_1 - \varepsilon_2} - x_{\varepsilon_1} - x_{\varepsilon_1 + \varepsilon_2}$$

is an extremal element. Let C be the Lie subalgebra of $\mathfrak{g}$ generated by M and d.

We show that it is $\mathfrak{g}$. Because $d + M = x_{\varepsilon_1} + M$, we see $x_{\varepsilon_1} \in C$; whence $V_1 \cap C \neq \{0\}$. By the irreducibility of V_1, we find that $x_{\pm\varepsilon_j} \in C$ for all j. Thus, C contains all root elements of the standard Chevalley basis $\mathscr{B}$ of $\mathfrak{g}$ and so coincides with $\mathfrak{g}$.

The conclusion is that $t(B_n) \leq t(D_n) + 1$. In particular, the proof of the theorem for B_n is complete once the theorem is shown to hold for D_n (as then $t(D_n) = n$).

8.3. *Type C_n.* Suppose $n \geq 2$ and let $\mathfrak{g}$ be the Lie algebra of type C_n. Take M to be the Lie subalgebra of type C_{n-1} generated by the root elements with roots in the linear span of $\varepsilon_2, \ldots, \varepsilon_n$. Then $\mathfrak{g}$ decomposes as

$$\mathfrak{g} = M \oplus V_1 \oplus V_{-1} \oplus V_2 \oplus V_{-2} \oplus V_0,$$

where V_1 and V_{-1} are natural M-modules, spanned by all root elements with first simple root coordinate 1, respectively -1, and V_2, V_0, V_{-2} are trivial modules, spanned by $x_{2\varepsilon_1}$, $h_{2\varepsilon_1}$, and $x_{-2\varepsilon_1}$, respectively.

Now take

$$d_1 = \exp(x_{-(\varepsilon_1+\varepsilon_2)}) x_{2\varepsilon_1} = x_{2\varepsilon_1} - x_{\varepsilon_1 - \varepsilon_2} - x_{-2\varepsilon_2}$$

and

$$d_2 = \exp(x_{\varepsilon_1+\varepsilon_2}) x_{-2\varepsilon_1} = x_{-2\varepsilon_1} + x_{-(\varepsilon_1 - \varepsilon_2)} - x_{2\varepsilon_2}.$$

Let C be the Lie subalgebra generated by M, d_1, and d_2. Then $d_1 + M = x_{2\varepsilon_1} - x_{\varepsilon_1 - \varepsilon_2} + M$ and $d_2 + M = x_{-2\varepsilon_1} + x_{-(\varepsilon_1 - \varepsilon_2)} + M$.

Now $C = [x_{-2\varepsilon_2}, d_2] + C = x_{-(\varepsilon_1+\varepsilon_2)} + C$, and $C = [x_{2\varepsilon_2}, d_1] + C = x_{\varepsilon_1+\varepsilon_2} + C$; so generators for each natural M-submodule of $\mathfrak{g}$ are in C. But then $C = d_1 + C = x_{2\varepsilon_1} + C$ and $C = d_2 + C = x_{-2\varepsilon_1} + C$, so V_2 and V_{-2} are also contained in C. We have established that $\mathfrak{g} \subseteq C$, whence $C = \mathfrak{g}$.

The conclusion is that $t(\mathfrak{g}) \leq t(M) + 2$. If $n = 2$, then M is of type A_1, so we have $t(M) = 2$ and $t(\mathfrak{g}) \leq 4$. By induction, we find $t(\mathfrak{g}) \leq 2n$ for arbitrary n.

8.4. *Type* D_n. First we consider the case $n = 4$. Let M be the subalgebra of $\mathfrak{g}$ generated by the root elements corresponding to the roots $x_{\pm(\varepsilon_1-\varepsilon_2)}$, $x_{\pm(\varepsilon_2-\varepsilon_3)}$, and $x_{\pm(\varepsilon_1-\varepsilon_3)}$. Then M is of type A_2 and $\mathfrak{g}$ decomposes as

$$\mathfrak{g} = M \oplus V_{\varepsilon_3-\varepsilon_4} \oplus V_{\varepsilon_3+\varepsilon_4} \oplus V_{1,0} \oplus V_{0,1} \oplus V_{1,1} \oplus V_{-1,0} \oplus V_{0,-1} \oplus V_{-1,-1},$$

where $V_{i,j}$ is a three-dimensional module linearly spanned by the root elements corresponding to simple roots of the form $\lambda(\varepsilon_1 - \varepsilon_2) + \mu(\varepsilon_2 - \varepsilon_3) + i(\varepsilon_3 - \varepsilon_4) + j(\varepsilon_3 + \varepsilon_4)$ where λ, μ are scalars and $V_{\varepsilon_3-\varepsilon_4}$ and $V_{\varepsilon_3+\varepsilon_4}$ are one-dimensional M-modules.

Consider

$$\begin{aligned} d &= \exp(x_{\varepsilon_1+\varepsilon_4}) \exp(x_{-(\varepsilon_3-\varepsilon_4)}) \exp(x_{-(\varepsilon_1+\varepsilon_3)}) x_{\varepsilon_3-\varepsilon_4} \\ &= -x_{\varepsilon_1+\varepsilon_3} - 2x_{\varepsilon_1+\varepsilon_4} + x_{\varepsilon_3-\varepsilon_4} - 2x_{-(\varepsilon_3-\varepsilon_4)} \\ &\quad -x_{-(\varepsilon_1+\varepsilon_3)} + x_{-(\varepsilon_1+\varepsilon_4)} + h_{-(\varepsilon_3-\varepsilon_4)} + h_{\varepsilon_1+\varepsilon_4}. \end{aligned}$$

This is an extremal element as it is the image of a root element under an automorphism of $\mathfrak{g}$.

Denote by C the Lie subalgebra of $\mathfrak{g}$ generated by M and d. A computation shows that

$$[x_{-(\varepsilon_1-\varepsilon_3)}, [x_{\varepsilon_1-\varepsilon_2}, d]] = -x_{-(\varepsilon_2-\varepsilon_3)} + x_{-(\varepsilon_1+\varepsilon_2)}.$$

Since $x_{-(\varepsilon_1-\varepsilon_3)}$, $x_{\varepsilon_1-\varepsilon_2}$, $x_{-(\varepsilon_2-\varepsilon_3)} \in M$, we derive $x_{-(\alpha_1+2\alpha_2+\alpha_3+\alpha_4)} = x_{-(\varepsilon_1+\varepsilon_2)} \in C$; that is, the lowest root element belongs to C. It follows that $V_{-1,-1} \subseteq C$.

Another computation yields

$$[x_{-(\varepsilon_2-\varepsilon_3)}, [x_{-(\varepsilon_1-\varepsilon_2)}, d]] = -2x_{\varepsilon_3+\varepsilon_4} + x_{-(\varepsilon_1-\varepsilon_3)}.$$

Similar to the above, we derive $x_{\alpha_4} = x_{\varepsilon_3+\varepsilon_4} \in C$. So $V_{0,1} \subseteq C$.

By computation, we have

$$[x_{\varepsilon_1-\varepsilon_2}, [h_{\varepsilon_1-\varepsilon_3}, [x_{-(\varepsilon_1+\varepsilon_3)}, d]]] = x_{-(\varepsilon_2+\varepsilon_4)}.$$

Since $x_{\varepsilon_1-\varepsilon_2}, h_{\varepsilon_1-\varepsilon_3} \in M$ and $x_{-(\varepsilon_1+\varepsilon_3)} \in V_{-1,-1} \subset C$, we see that $x_{-(\alpha_2+\alpha_4)} = x_{-(\varepsilon_2+\varepsilon_4)} \in C$, whence $V_{0,-1} \subset C$.

Since $x_{-(\varepsilon_1+\varepsilon_4)} \in V_{0,-1} \subseteq C$ and $h_{\varepsilon_1-\varepsilon_3} \in M$, we have

$$\begin{aligned} C &= [x_{-(\varepsilon_1-\varepsilon_3)}, [x_{\varepsilon_1-\varepsilon_3}, d]] + C \\ &= x_{-(\varepsilon_1+\varepsilon_4)} + 2h_{\varepsilon_1-\varepsilon_3} + x_{\varepsilon_3-\varepsilon_4} + C = x_{\varepsilon_3-\varepsilon_4} + C, \end{aligned}$$

and so $x_{\alpha_3} = x_{\varepsilon_3-\varepsilon_4} \in C$. So $V_{1,0} \subseteq C$.

We have seen that the root elements corresponding to all four simple roots and to the lowest root belong to C. Since they generate $\mathfrak{g}$, we conclude that $C = \mathfrak{g}$; whence $t(\mathfrak{g}) \le t(M) + 1 = 4$.

Assume, from now on, that $n > 4$. We show that the Lie algebra $\mathfrak{g}$ of type D_n satisfies $t(\mathfrak{g}) \le n$.

Let M be the Lie subalgebra of $\mathfrak{g}$ generated by all root elements corresponding to the subrootsystem on $\varepsilon_i \pm \varepsilon_j$ for $2 \le i < j \le n$. Then M has type D_{n-1}. As an M-module, $\mathfrak{g}$ decomposes as

$$\mathfrak{g} = M \oplus V_1 \oplus V_{-1} \oplus H_0,$$

where H_0 is a one-dimensional toral subalgebra centralizing M, and V_1, V_{-1} are natural modules spanned by all root elements whose root vectors have first coordinate (with respect to the basis of simple roots) equal to 1, or -1, respectively.

By induction $t(M) \le n - 1$, so we need to find an extremal element of $\mathfrak{g}$ that, together with M, generates $\mathfrak{g}$. Take

$$d = \exp(x_{-(\varepsilon_1-\varepsilon_2)})x_{\varepsilon_1-\varepsilon_2} = x_{\varepsilon_1-\varepsilon_2} - h_{\varepsilon_1-\varepsilon_2} - x_{-(\varepsilon_1-\varepsilon_2)}.$$

Now $[x_{\varepsilon_2-\varepsilon_3}, d] + M = -x_{\varepsilon_1-\varepsilon_3} + M$, so $x_{\varepsilon_1-\varepsilon_3} \in C$; whence every root element whose root has first root coordinate 1 belongs to C. That is, $V_1 \subseteq C$ and therefore $x_{\varepsilon_1-\varepsilon_2} \in C$. Similarly, $C = [x_{-(\varepsilon_2-\varepsilon_3)}, d] + C = -x_{-(\varepsilon_1-\varepsilon_3)} + C$, and so also every root element whose root has first coordinate -1 belongs to C. That is, $V_{-1} \subseteq C$ and therefore $x_{-(\varepsilon_1+\varepsilon_2)} \in C$. Thus, all root elements from the standard Chevalley basis belong to C, proving $C = \mathfrak{g}$, and $t(\mathfrak{g}) \le t(M) + 1 = n$.

8.5. *Type E_6.* Let us denote by $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6$ the simple roots of $\mathfrak{e}_6$. Let M be a subalgebra of $\mathfrak{e}_6$ generated by $M = \langle x_{\alpha_2}, x_{\alpha_3}, x_{\alpha_4}, x_{\alpha_5}, x_{-(\alpha_2+\alpha_3+2\alpha_4+\alpha_5)}\rangle$. Then M is of type D_4. One has the decomposition of $\mathfrak{e}_6$

$$\mathfrak{e}_6 = M \oplus W \oplus V_{0,1} \oplus V_{1,0} \oplus V_{1,1} \oplus V_{0,-1} \oplus V_{-1,0} \oplus V_{-1,-1},$$

where $V_{a,b}$ is the module generated by all x_α with a the coefficient of α_1 and b the coefficient of α_6 in α. The action of M on W is trivial.

We need one more extremal element.

$$\begin{aligned} d &= \exp(x_{-(\alpha_1+\alpha_3+\alpha_4)})\exp(x_{\alpha_3+\alpha_4+\alpha_5+\alpha_6})\exp(x_{-(\alpha_1+\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+\alpha_6)})x_{\alpha_1} \\ &= x_{\alpha_1} - x_{-(\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+\alpha_6)} + x_{\alpha_1+\alpha_3+\alpha_4+\alpha_5+\alpha_6} + x_{-(\alpha_2+\alpha_4+\alpha_5)} \\ &\quad -x_{-(\alpha_3+\alpha_4)} - x_{-(\alpha_1+\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6)} \\ &\quad +x_{\alpha_5+\alpha_6} - x_{-(\alpha_1+\alpha_2+\alpha_3+2\alpha_4+\alpha_5)}. \end{aligned}$$

We claim that the subalgebra C generated by M and d is the whole algebra $\mathfrak{e}_6$.

The root elements $x_{-(\alpha_2+\alpha_4+\alpha_5)}$ and $x_{-(\alpha_3+\alpha_4)}$ belong to M, so $e := d - x_{-(\alpha_2+\alpha_4+\alpha_5)} + x_{-(\alpha_3+\alpha_4)}$ is in C. Since x_{α_3}, $x_{-\alpha_5}$, and $x_{-\alpha_2}$ are in M, the following brackets are in C:

$$[x_{\alpha_3}, e] = x_{\alpha_1+\alpha_3},\ [x_{-\alpha_5}, e] = x_{\alpha_6},\ \text{and}$$
$$[x_{-\alpha_2}, e] = -x_{-(\alpha_1+2\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6)}.$$

But $x_{\alpha_1+\alpha_3} \in C$ implies $V_{1,0} \subset C$, and so $x_{\alpha_1} \in C$.

Now C contains all the simple root elements and the lowest root element. By Lemma 8.1, we are done.

8.6. *Type E_7.* Let us denote by $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7$ the simple roots of $\mathfrak{e}_7$. Take the subalgebra M of type D_4 in $\mathfrak{e}_7$ exactly as in $\mathfrak{e}_6$.

To generate the whole algebra $\mathfrak{e}_7$, consider the extremal element

$$\begin{aligned}
d = {} & \exp(x_{\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7}) \exp(x_{-(\alpha_1+\alpha_3)}) \\
& \exp(x_{-(\alpha_1+\alpha_2+\alpha_3+2\alpha_4+\alpha_5+\alpha_6)}) \exp(x_{\alpha_3+\alpha_4+\alpha_5+\alpha_6}) \\
& \exp(x_{-(\alpha_1+\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+\alpha_6+\alpha_7)}) x_{\alpha_1} \\
= {} & x_{\alpha_1} + x_{\alpha_4+\alpha_5+\alpha_6} + x_{\alpha_1+\alpha_3+\alpha_4+\alpha_5+\alpha_6} + x_{\alpha_2+\alpha_4+\alpha_5+\alpha_6+\alpha_7} \\
& + x_{\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7} - x_{\alpha_1+\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6} \\
& + x_{\alpha_1+2\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6+\alpha_7} - x_{-\alpha_3} + x_{-(\alpha_2+\alpha_4)} - x_{-(\alpha_4+\alpha_5)} \\
& - x_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)} + x_{-(\alpha_1+\alpha_3+\alpha_4+\alpha_5)} - x_{-(\alpha_2+\alpha_3+2\alpha_4+\alpha_5+\alpha_6)} \\
& - x_{-(\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+\alpha_5+\alpha_6)} - x_{-(\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+\alpha_6+\alpha_7)} \\
& - x_{-(\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+2\alpha_5+\alpha_6+\alpha_7)} - x_{-(\alpha_1+2\alpha_2+2\alpha_3+4\alpha_4+3\alpha_5+2\alpha_6+\alpha_7)} \\
& + x_{-(2\alpha_1+2\alpha_2+3\alpha_3+4\alpha_4+3\alpha_5+2\alpha_6+\alpha_7)}.
\end{aligned}$$

We show that the subalgebra C generated by M and d is the whole algebra $\mathfrak{e}_7$. The root elements $x_{-\alpha_3}$, $x_{-(\alpha_2+\alpha_4)}$, and $x_{-(\alpha_4\alpha_5)}$ belong to M, so $e := d + x_{-\alpha_3} - x_{-(\alpha_2+\alpha_4)} + x_{-(\alpha_4+\alpha_5)}$ is in C. The Lie algebra $\mathfrak{e}_7$ decomposes as

$$\begin{aligned}
\mathfrak{e}_7 = {} & M \oplus W \oplus V_{0,0,1} \oplus V_{0,1,0} \oplus V_{0,1,1} \oplus V_{0,2,1} \oplus V_{1,0,0} \oplus V_{1,1,0} \\
& \oplus V_{1,1,1} \oplus V_{1,2,1} \oplus V_{2,2,1} \oplus V_{0,0,-1} \\
& \oplus V_{0,-1,0} \oplus V_{0,-1,-1} \oplus V_{0,-2,-1} \\
& \oplus V_{-1,0,0} \oplus V_{-1,-1,0} \oplus V_{-1,-1,-1} \\
& \oplus V_{-1,-2,-1} \oplus V_{-2,-2,-1},
\end{aligned}$$

where W is a three-dimensional module with trivial M action. The modules $V_{1,0,0}$, $V_{0,1,1}$, $V_{0,1,0}$, and $V_{1,2,1}$ have dimension 8 and $V_{a,b,c}$ is generated by all root elements not in M such that a is the coefficient of α_1, b of α_6, and c of α_7.

The bracket $[x_{\alpha_2+\alpha_3+2\alpha_4+\alpha_5}, [x_{\alpha_3}, e]]$ is easily computed to be a nonzero scalar multiple of $x_{\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+\alpha_5}$, and, since both arguments of the bracket belong to M, the element $x_{\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+\alpha_5}$ belongs to C. Consequently, $V_{1,0,0}$ is contained in C and so $x_{\alpha_1} \in C$.

Since $x_{-(\alpha_4+\alpha_5)}, x_{\alpha_5} \in C$, we have $[x_{\alpha_5}, [x_{-(\alpha_4+\alpha_5)}, e]] = -x_{\alpha_5+\alpha_6} \in C$. So $V_{0,1,0} \subset C$ and $x_{\alpha_6} \in C$. Also, $[x_{-\alpha_5}, [x_{\alpha_3}, e]]$ belongs to C. Expanding it, we find $x_{-(\alpha_1+\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+\alpha_6)} \in C$. Hence C contains $V_{-1,-1,0}$ and taking brackets with x_{α_1} we obtain $V_{0,-1,0} \subset C$.

Now $x_{\alpha_7} \in C$, as $[x_{-(\alpha_3+\alpha_4+\alpha_5+\alpha_6)}, [x_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6)}, e]]$ is a nonzero element of $kx_{\alpha_7} + V_{0,-1,0}$ which also belongs to C.

Finally, as $x_{\alpha_5+\alpha_6} \in C$ and $[x_{-\alpha_4}, e] = x_{\alpha_5+\alpha_6} - x_{-(\alpha_1+\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6+\alpha_7)}$, we find $V_{-1,-1,-1} \subset C$. Taking brackets with a convenient element of $V_{-1,-1,0}$, we see that also the lowest root element is in C. Now $C = \mathfrak{e}_7$ by Lemma 8.1.

8.7. *Type E_8.* Let $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8$ be the simple roots of $\mathfrak{e}_8$. Adopt here the notation $(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8)$ for a root $\alpha = \sum_{i=1}^{8} a_i\alpha_i$. Consider the subalgebra M of type D_4 in $\mathfrak{e}_8$ as in $\mathfrak{e}_6$. We have the following decomposition of $\mathfrak{e}_8$ into M modules.

$$\mathfrak{e}_8 = M \oplus W \oplus \bigoplus_{a,k,l,m \in I} V_{a,k,l,m},$$

where W is a 28-dimensional trivial module, $V_{a,k,l,m}$ is an irreducible 8-dimensional module generated by all roots elements x_α where a, k, l, m are respectively the coefficients of α_1, α_6, α_7, α_8 in α, and

$$\begin{aligned} I = \pm\{&(0,1,0,0), (0,1,1,0), (0,1,1,1), (1,0,0,0), (1,1,0,0),\\ &(1,1,1,0), (1,1,1,1), (1,2,1,0), (1,2,1,1), (1,2,2,1),\\ &(1,3,2,1), (2,3,2,1)\}. \end{aligned}$$

To generate the whole algebra $\mathfrak{e}_8$, consider the extremal element

$$\begin{aligned} d &= \exp(x_{01111110})\exp(x_{-11121110})\exp(x_{01122111})\exp(x_{-10111111})\\ &\quad \exp(x_{12343321})\exp(x_{-23354321})x_{10000000}\\ &= x_{10000000} - x_{01011000} + x_{01111100}\\ &\quad + x_{00111111} + x_{11111110} + x_{11122111} - x_{11222211} - x_{12232210}\\ &\quad - x_{12233221} - x_{22343321} - x_{13354321} - x_{23465432} + x_{-10000000}\\ &\quad - x_{-00010000} - x_{-01011000} - x_{-00111111} + x_{-11121100} - x_{-01121110}\\ &\quad - x_{-11122111} - x_{-12232210} - x_{-11232221} + x_{-12243211} + x_{-22343321}\\ &\quad + x_{-13354321} - x_{-23465432}. \end{aligned}$$

Let C be the Lie subalgebra generated by M and d. Bracketing and using the decomposition of $\mathfrak{e}_8$ as an M-module, one can again derive equality between C and $\mathfrak{e}_8$. We omit further details, as they are very similar to the other cases dealt with (but lengthier).

8.8. *Type* F_4. Write α_1, α_2, α_3, and α_4 for the simple roots of $\mathfrak{f}_4$. So, $\alpha_1 = \varepsilon_2 - \varepsilon_3$, $\alpha_2 = \varepsilon_3 - \varepsilon_4$, $\alpha_3 = \varepsilon_4$, and $\alpha_4 = \frac{1}{2}(\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4)$.

Let M be the subalgebra of $\mathfrak{f}_4$ generated by $x_{\alpha_1}, x_{\alpha_2}, x_{\alpha_2+2\alpha_3}, x_{\alpha_2+2\alpha_3+2\alpha_4}$, and $x_{-(2\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4)}$. That is, $M = \langle x_{\varepsilon_1-\varepsilon_2}, x_{\varepsilon_2-\varepsilon_3}, x_{\varepsilon_3-\varepsilon_4}, x_{\varepsilon_3+\varepsilon_4}, x_{-(\varepsilon_1+\varepsilon_2)}\rangle$. According to Lemma 8.1, M is of type D_4. By 8.4 we know then that M is generated by four extremal elements.

Define the following element in $\mathfrak{f}_4$:

$$d = \exp(x_{-(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4)})\exp(x_{\alpha_4})(x_{\alpha_2+2\alpha_3}) = x_{\alpha_2+2\alpha_3} - x_{\alpha_2+2\alpha_3+\alpha_4}$$
$$- x_{\alpha_2+2\alpha_3+2\alpha_4} - x_{-(\alpha_1+\alpha_2+\alpha_3)} - x_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)} - x_{-(2\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4)}.$$

The element d is extremal since $\alpha_2 + 2\alpha_3$ is long. Consider the subalgebra C generated by M and d.

The root elements $x_{\alpha_2+2\alpha_3}$, $x_{\alpha_2+2\alpha_3+2\alpha_4}$, and $x_{-(2\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4)}$ are in M. So $f := d - x_{\alpha_2+2\alpha_3} + x_{\alpha_2+2\alpha_3+2\alpha_4} + x_{-(2\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4)} = -x_{\alpha_2+2\alpha_3+\alpha_4} - x_{-(\alpha_1+\alpha_2+\alpha_3)} - x_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)}$ is in C.

The Lie algebra $\mathfrak{f}_4$ decomposes as

$$\mathfrak{f}_4 = M \oplus V_{0,1} \oplus V_{1,0} \oplus V_{0,0} \oplus V_{1,1} \oplus V_{1,2}$$
$$\oplus V_{0,-1} \oplus V_{-1,0} \oplus W_{0,0} \oplus V_{-1,-1} \oplus V_{-1,-2},$$

where $V_{0,1}$ is the module generated by x_{α_4}, $x_{\alpha_3+\alpha_4}$, $x_{\alpha_2+\alpha_3+\alpha_4}$, and $x_{\alpha_2+2\alpha_3+\alpha_4}$, and $V_{1,0} = \langle x_{\alpha_1+\alpha_2+\alpha_3}\rangle$, $V_{0,0} = \langle x_{\alpha_3}, x_{\alpha_2+\alpha_3}\rangle$, $V_{1,1} = \langle x_{\alpha_1+\alpha_2+\alpha_3+\alpha_4}, x_{\alpha_1+\alpha_2+2\alpha_3+\alpha_4}, x_{\alpha_1+2\alpha_2+2\alpha_3+\alpha_4}, x_{\alpha_1+2\alpha_2+3\alpha_3+\alpha_4}\rangle$, and $V_{1,2} = \langle x_{\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4}\rangle$. The module $W_{0,0}$ is generated by $x_{-\alpha_3}$, $x_{-(\alpha_2+\alpha_3)}$, and obviously the remaining modules are generated by the negative root elements.

We have $[x_{\alpha_1+\alpha_2+2\alpha_3}, f] + C = x_{\alpha_3} + C$ since $x_{\alpha_1+\alpha_2+2\alpha_3} \in M$. Hence x_{α_3} is in C and then $V_{0,0} \subset C$.

Now the bracket of $x_{\alpha_1+\alpha_2+2\alpha_3+2\alpha_4} \in M$ with f leads to the conclusion that $x_{\alpha_3+\alpha_4} \in C$, thus $V_{0,1} \subset C$. In particular, $x_{\alpha_4} \in C$.

By Lemma 8.1, the algebra $\mathfrak{f}_4$ is generated by x_{α_1}, x_{α_2}, x_{α_3}, x_{α_4}, and $x_{-(2\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4)}$. But C contains all these root elements. Therefore $C = \mathfrak{f}_4$. The conclusion is that $t(\mathfrak{f}_4) \leq t(D_4) + 1 = 5$.

8.9. *Type* G_2. Let α and β be the simple roots of $\mathfrak{g}_2$. Let M be the Lie subalgebra of $\mathfrak{g}_2$ generated by the long root elements x_β, $x_{3\alpha+\beta}$, $x_{-(3\alpha+2\beta)}$. This subalgebra is of type A_2.

Consider the extremal element $d = \exp(x_{-(2\alpha+\beta)})(x_{3\alpha+2\beta})$; it is extremal since $3\alpha + 2\beta$ is a long root. We have

$$d = x_{3\alpha+2\beta} + x_{\alpha+\beta} - x_{-\alpha}.$$

Let C be the subalgebra of $\mathfrak{g}_2$ generated by M and d. We will prove that C is actually $\mathfrak{g}_2$. Since x_α, x_β, and $x_{-(3\alpha+2\beta)}$ generate $\mathfrak{g}_2$, and the latter two are obviously in C, it is enough to show that x_α is also in C.

Since $d - x_{3\alpha+2\beta} \in C$, so is the bracket

$$[x_{-(3\alpha+2\beta)}, d - x_{3\alpha+2\beta}] = [x_{-(3\alpha+2\beta)}, x_{\alpha+\beta} - x_{-\alpha}] = -x_{-(2\alpha+\beta)}.$$

Therefore $[x_{3\alpha+\beta}, x_{-(2\alpha+\beta)}] = x_\alpha$ is also in C.

As before we conclude that $C = \mathfrak{g}_2$ and so $\mathfrak{g}_2$ is generated by four extremal elements.

9. THE BILINEAR FORM AND THE KILLING FORM

Throughout this section, L is a finite dimensional Lie algebra (over a field k of characteristic not 2) generated by extremal elements. Recall that the Killing form κ is defined by $\kappa(x, y) := \mathrm{tr}(\mathrm{ad}_x \mathrm{ad}_y)$ for $x, y \in L$. We consider connections between the associative bilinear form f defined in Theorem 2.5 and the Killing form κ.

Fix $x \in \mathcal{E}$ and $y \in L$. Set $\varphi := \mathrm{ad}_x \mathrm{ad}_y$.

LEMMA 9.1. *We have* $\varphi^2 + \frac{1}{2}f(x, y)\varphi : L \to kx + k[x, y]$.

Proof. For $z \in L$, using Lemma 2.2, we find

$$\varphi^2(z) = [x, [y, [x, [y, z]]]] = \frac{1}{2}(f(x, [y, [y, z]])x - f(x, [y, z])[x, y] - f(x, y)[x, [y, z]]).$$

This implies that $\varphi^2 + \frac{1}{2}f(x, y)\varphi$ maps L to $kx + k[x, y]$. ∎

LEMMA 9.2. *Let* $x \in \mathcal{E}$, $y \in L$. *Then* φ *satisfies the following properties.*

(a) *If* $f(x, y) = 0$, *then all eigenvalues of* φ *are* 0. *In particular,* $\kappa(x, y) = 0$.

(b) *If* $f(x, y) = -2$, *then* φ *has eigenvalue* 2 *with multiplicity* 2, *eigenvalue* 1 *with multiplicity* $s - 2$, *and the remaining eigenvalues are* 0. *Here* $s = \dim \mathrm{ad}_x(L)$. *In particular,* $\kappa(x, y) = s + 2$.

Proof. Note that $\varphi(x) = -f(x, y)x$ and (by Lemma 2.2) $\varphi([x, y]) = -f(x, y)[x, y]$. Hence $U = kx + k[x, y]$ is invariant under φ. Furthermore, $\varphi^2 + \frac{1}{2}f(x, y)\varphi$ is zero on L/U by Lemma 9.1. We write φ on L/U in Jordan normal form with eigenvalues 0 or $-\frac{1}{2}f(x, y)$. Now (a) is obvious. Next suppose that $f(x, y) = -2$. Then $x, [x, y]$ are linearly independent. Denote by E the eigenspace of φ (with eigenvalue 1). Then $E + U/U = \mathrm{ad}_x(L)/U$ and $\dim E = \dim \mathrm{ad}_x(L) - 2$. This proves the lemma. ∎

We now consider connections between the radicals of the two forms. Recall that $\mathrm{Rad}(f)$ denotes the radical of f; similarly, we write $\mathrm{Rad}(\kappa)$ for the radical of κ.

COROLLARY 9.3. *We have* $\mathrm{Rad}(f) \subseteq \mathrm{Rad}(\kappa)$.

Proof. Let $y \in \mathrm{Rad}(f)$. Then $f(x, y) = 0$ for all $x \in \mathscr{E}$. Hence $\kappa(x, y) = 0$ for all $x \in \mathscr{E}$ by Lemma 9.2. Therefore $y \in \mathrm{Rad}(\kappa)$. ∎

Remark 9.4. Lemma 9.2 gives a way to tell if one of the Lie algebras of Chevalley type is the radical of its Killing form. (In this case $\mathrm{Rad}(f)$ is properly contained in $\mathrm{Rad}(\kappa)$.) For a root α, set $\gamma_\alpha = \dim \mathrm{ad}_{x_\alpha}(L) - 2$. Then by Lemma 9.2, $\kappa(x_\alpha, x_{-\alpha}) = \gamma_\alpha + 4$. When there is only one root length, γ_α is the number of roots which have inner product -1 with α. We look at the case of $\mathfrak{sl}_3$. For $\alpha = e_i - e_j$, there are exactly two such roots, $e_k - e_i$, and $e_j - e_k$, so $\kappa(x_\alpha, x_{-\alpha}) = 2 + 4 = 6$. Hence $\mathfrak{sl}_3$ has Killing form zero if k is of characteristic 3. Similarly, for E_8, we have $\kappa(x_\alpha, x_{-\alpha}) = 60$, and so the Killing form is identically zero in characteristics 3 and 5.

LEMMA 9.5. *If* $\mathrm{char}(k) = 0$, *then* $\mathrm{Rad}(f) = \mathrm{Rad}(\kappa)$.

Proof. One inclusion comes from Corollary 9.3. Let $y \in \mathrm{Rad}(\kappa)$. Assume that $y \notin \mathrm{Rad}(f)$. Then there exists $x \in \mathscr{E}$ such that $f(x, y) = -2$. Hence $\kappa(x, y) = \dim \mathrm{ad}_x(L) + 2 \neq 0$ in characteristic 0. This is a contradiction. ∎

In what follows we determine some more properties about the radicals of the Killing form and of f.

LEMMA 9.6. *Let J be an ideal of L and* $N := \mathrm{span}_k\{x \in \mathscr{E} \mid x \notin J\}$. *Then* $f(N, J) = 0$.

Proof. It is enough to show for $x \in \mathscr{E} \setminus J$ and $y \in J$ that $f(x, y) = 0$. We have $f(x, y)x = [x, [x, y]] \in J$. Hence $f(x, y) = 0$. ∎

LEMMA 9.7. *Suppose that K is a solvable ideal of L. Then* $\mathscr{E} \cap K \subseteq \mathrm{Rad}(f)$.

Proof. Let $x \in \mathscr{E} \cap K$. Take $y \in \mathscr{E}$. If $y \notin K$, then $f(x, y) = 0$ by Lemma 9.6. If $y \in K$, suppose $f(x, y) \neq 0$. Then $\langle x, y\rangle \simeq \mathfrak{sl}_2$. Hence $\langle x, y\rangle$ is a non-solvable Lie subalgebra of K which is a contradiction. So $f(x, y) = 0$.

This means that $f(x, y) = 0$ for all $x \in \mathscr{E} \cap K$ and $y \in \mathscr{E}$. The lemma follows as $\mathscr{E}$ linearly spans L; cf. Lemma 2.4. ∎

Let $\mathrm{Rad}(L)$ be the maximal solvable ideal in L.

PROPOSITION 9.8. *We have* $\mathrm{Rad}(L) \subseteq \mathrm{Rad}(f)$.

Proof. Suppose that K is a solvable ideal of L and let $x \in K$. For $y \in \mathcal{E}$, either $y \in \mathcal{E} \setminus K$, in which case $f(x, y) = 0$ by Lemma 9.6, or $y \in \mathcal{E} \cap K$, and then $f(x, y) = 0$ by Lemma 9.7. ∎

Remark 9.9. An example where $\mathrm{Rad}(L)$ is properly contained in $\mathrm{Rad}(f)$ is the 14-dimensional Lie algebra L of type G_2 over a field k of characteristic 3. Then, $\mathrm{Rad}(L) = 0$ but $\mathrm{Rad}(f)$ is a 7-dimensional ideal, generated by the short root elements of a Chevalley basis. Proposition 9.12 shows that characteristic 3 is indeed exceptional.

In fact, $\mathrm{Rad}(f)$ and $\mathrm{Rad}(L)$ are just two ideals of a chain of five. Denote by $\mathrm{SanRad}(L)$ the linear span of all *sandwiches* of $\mathcal{E}$, that is, those $x \in \mathcal{E}$ for which $\mathrm{ad}_x^2 = 0$. We claim that $\mathrm{SanRad}(L)$ is an ideal of L. Indeed, if $x \in \mathrm{SanRad}(L) \cap \mathcal{E}$ and $y \in \mathcal{E}$, then $f_x = 0$, so Corollary 2.6 shows that either $[x, y] = 0$ or $[x, y] \in \mathcal{E}$ with $f_{[x,y]} = 0$, that is, $[x, y] \in \mathrm{SanRad}(L)$. Since the restriction of f to $\mathrm{SanRad}(L)$ vanishes, the latter is a nilpotent Lie subalgebra of L by Lemma 4.2, so $\mathrm{SanRad}(L)$ is contained in $\mathrm{NilRad}(L)$, the nilpotent radical of L. We thus have the chain

$$\mathrm{SanRad}(L) \subseteq \mathrm{NilRad}(L) \subseteq \mathrm{Rad}(L) \subseteq \mathrm{Rad}(f) \subseteq \mathrm{Rad}(\kappa).$$

We know that the last two inclusions may be proper, and, in view of Case (2) of Theorem 5.2, we have examples where $\mathrm{SanRad}(L)$ is strictly contained in $\mathrm{Rad}(L)$.

LEMMA 9.10. *Suppose $x \in \mathcal{E} \setminus \mathrm{Rad}(f)$ and $y \in \mathrm{Rad}(f)$. Then $[x, y]$ satisfies $\mathrm{ad}_{[x,y]}^4 = 0$.*

Proof. Put $X = \mathrm{ad}_x$ and $Y = \mathrm{ad}_y$. Then, by (3) of Lemma 2.2, for $z \in L$,

$$\begin{aligned} 2XYXYz &= 2[x, [y, [x, [y, z]]]] \\ &= f(x, [y, [y, z]])x - f(x, [y, z])[x, y] - f(x, y)[x, [y, z]] = 0. \end{aligned}$$

For the last equality, use that f is associative and $y \in \mathrm{Rad}(f)$. We obtain $XYXY = 0$. Also, $X^2Y = 0$, as $X^2Y(L) \subseteq \mathrm{Rad}(f) \cap kx = 0$. Consequently,

$$\begin{aligned} \mathrm{ad}_{[x,y]}^4 &= (XY - YX)^4 = (XYXY - XY^2X - YX^2Y + YXYX)^2 \\ &= (YXYX - XY^2X)^2 \\ &= XY^2X^2Y^2X + (YX)^4 - XY^2(XY)^2X - YXYX^2Y^2X = 0. \end{aligned}$$ ∎

LEMMA 9.11. *Let K be an ideal of L which is contained in Rad(f). Then $\overline{L} := L/K$ is linearly spanned by extremal elements with induced form $\overline{f}$ defined by $\overline{f}(\overline{x}, \overline{y}) := f(x, y)$ for $x, y \in L$.*

Proof. Since $K \subseteq \mathrm{Rad}(f)$, the expression $\overline{f}(\overline{x}, \overline{y}) := f(x, y)$ is well-defined for $x, y \in L$. For $x \in \mathcal{E}$, $y \in L$, $[\overline{x}, [\overline{x}, \overline{y}]] = \overline{f}(\overline{x}, \overline{y}) \cdot \overline{x}$, whence $\overline{x} \in \mathcal{E}(\overline{L})$. ∎

We owe the proof of the following result to Gabor Ivanyos.

PROPOSITION 9.12. *If the characteristic of k is not* 2 *or* 3, *then* $\mathrm{Rad}(f) = \mathrm{Rad}(L)$.

Proof. Recall that $\mathrm{Rad}(L) \subseteq \mathrm{Rad}(f)$, so if $\mathrm{Rad}(f)$ is contained in the center $Z(L)$ of L, there is nothing to prove. Suppose, therefore, $\mathrm{Rad}(f) \nsubseteq Z(L)$. We show that $\mathrm{SanRad}(L) \neq 0$.

By Lemma 9.10, there is a nonzero element $y \in \mathrm{Rad}(f)$ with $\mathrm{ad}_y^4 = 0$. (If $\mathcal{E} \cap \mathrm{Rad}(f) \neq \emptyset$, any element of the intersection will do; otherwise, take $y \in \mathrm{Rad}(f)$, and $x \in \mathcal{E}$ such that $[x, y] \neq 0$, whose existence is guaranteed by the hypotheses that $L = \langle \mathcal{E} \rangle$ and $\mathrm{Rad}(f) \nsubseteq Z(L)$, and apply Lemma 9.10.) By Proposition 2.1.5 of [7] (see also Proposition 1.5 of [1]), if the characteristic of k is not 2 or 3, the element $z = \mathrm{ad}_y^3(x)$ for any $x \in L$ satisfies $\mathrm{ad}_z^3 = 0$. In particular, there is a nonzero element $y \in \mathrm{Rad}(f)$ with $\mathrm{ad}_y^3 = 0$.

If $\mathrm{ad}_y^2 = 0$, then we are done. Otherwise, there is $b \in \mathcal{E}$ with $x = \mathrm{ad}_y^2(b) \neq 0$.

If $b \in \mathrm{Rad}(f)$, then $b \in \mathrm{Rad}(f) \cap \mathcal{E} \subseteq \mathrm{SanRad}(L)$, and we are done. So, assume $b \notin \mathrm{Rad}(f)$. By Lemma 1.7(iii) of [1], as k has characteristic not 2 or 3, we have $\mathrm{ad}_x^2 = \mathrm{ad}_y^2 \mathrm{ad}_b^2 \mathrm{ad}_y^2$. Since $y \in \mathrm{Rad}(f)$ we have $\mathrm{ad}_y^2(L) \subseteq \mathrm{Rad}(f)$, and since $b \in \mathcal{E} \setminus \mathrm{Rad}(f)$ we have $\mathrm{ad}_b^2(\mathrm{Rad}(f)) = 0$; whence $\mathrm{ad}_x^2 = 0$, proving that $\mathcal{E} \cap \mathrm{Rad}(f) \neq 0$.This establishes $\mathrm{SanRad}(L) \neq 0$.

Thus, if $\mathrm{Rad}(f) \nsubseteq Z(L)$, it contains a nonzero sandwich, and so $\mathrm{NilRad}(L) \neq 0$. But then, in view of Lemma 9.11 and by induction on the dimension, $\mathrm{Rad}(f)$ is solvable. ∎

We finish this section with a result identifying $\mathrm{Rad}(f)$ and $\mathrm{Rad}(L)$ in arbitrary characteristic distinct from 2 under additional hypotheses on the structure of L. It shows that f plays a role similar to κ in the theory of Lie algebras of characteristic 0.

LEMMA 9.13. *Let $L = L_1 \oplus L_2$ be a direct sum of ideals. Then L_1 and L_2 are linearly spanned by extremal elements* (*with form f restricted to L_1 and L_2, respectively*). *Furthermore, L_1 and L_2 are orthogonal with respect to f.*

Proof. As $[L_1, L_2] = 0$, we have $\mathcal{E}(L_i) \subseteq \mathcal{E}$ for $i = 1, 2$. Let $x \in \mathcal{E}(L)$ and $y \in L$. Write $x = x_1 + x_2$, $y = y_1 + y_2$ with $x_1, y_1 \in L_1$, $x_2, y_2 \in L_2$. We calculate that $f(x, y)x_1 + f(x, y)x_2 = [x_1, [x_1, y]] + [x_2, [x_2, y]]$. Hence $[x_i, [x_i, y]] = f(x, y)x_i$ for all $y \in L$, and $x_i \in \mathcal{E}$ $(i = 1, 2)$.

Since L is linearly spanned by $\mathcal{E}$, each L_i is linearly spanned by the projections. Hence L_i is linearly spanned by extremal elements (with form f restricted to L_i).

Finally, for $z \in \mathscr{E}(L_1)$, $l_2 \in L_2$, we have $f(z, l_2)z = [z, [z, l_2]] = 0$. Since L_1 is linearly spanned by extremal elements, this shows that the decomposition $L = L_1 \oplus L_2$ is an orthogonal one with respect to f.

PROPOSITION 9.14. *We have* $\mathrm{Rad}(f) = 0$ *if and only if* L *is a direct sum of simple ideals*.

Proof. Assume that $L = L_1 \oplus L_2 \oplus \cdots \oplus L_r$ is a direct sum of simple ideals L_i. By Lemma 9.13, the decomposition is an orthogonal one with respect to f. Suppose that $L_i \subseteq \mathrm{Rad}(f)$ for some i. Then the form f restricted to L_i is trivial. But then L_i is nilpotent by Lemma 4.2, a contradiction. This means that the form f restricted to each L_i has a trivial radical. Hence $\mathrm{Rad}(f) = 0$.

As for the converse, assume $\mathrm{Rad}(f) = 0$. Suppose that I is a minimal nonzero ideal of L. By Proposition 9.8, I is not Abelian. Let J be the orthoplement of I with respect to f. As f is associative, J is an ideal of L. We claim that $L = I \oplus J$. For $I \cap J \neq 0$ would imply $I \subseteq J$ by the minimality of I, and hence $[I, J] = I$ as I is not Abelian; so $f(I, L) = f([I, J], L) = f(I, [J, L]) \subseteq f(I, J) = 0$, yielding the contradiction $I \subseteq \mathrm{Rad}(f)$.

Hence, if K is a nonzero ideal of I it also is an ideal of L, and so it coincides with I. This means that I is simple. Moreover, by Lemma 9.13, J is generated by extremal elements with the form $f|_J$. Note that $\mathrm{Rad}(f|_J) \subseteq \mathrm{Rad}(f) = 0$, so by induction on the dimension, we find that L is a direct sum of simple ideals. ∎

COROLLARY 9.15. *We have that* $L/\mathrm{Rad}(L)$ *is a direct sum of simple ideals if and only if* $\mathrm{Rad}(L) = \mathrm{Rad}(f)$.

Proof. In Lemma 9.8, we have already proved that $\mathrm{Rad}(L) \subseteq \mathrm{Rad}(f)$. Set $\overline{L} = L/\mathrm{Rad}(L)$ and let $\overline{f}$ be as in Lemma 9.11.

Since $\overline{\mathrm{Rad}(f)} = \mathrm{Rad}(\overline{f})$, we have $\mathrm{Rad}(L) = \mathrm{Rad}(f)$ if and only if $\mathrm{Rad}(\overline{f}) = 0$, so the corollary is a direct consequence of Proposition 9.14. ∎

10. ANALYSIS OF ROOT GROUPS

DEFINITION 10.1. For $y \in \mathscr{E}$, we define $U_y := \{\exp(y, t) \mid t \in k\}$ to be the root group associated to y. Since $\exp(cy, t) = \exp(y, ct)$ for all $c \in k$, the group only depends on the one-dimensional subspace ky.

By $\Sigma := \{U_y \mid y \in \mathscr{E}\}$ we denote the set of root groups associated to extremal elements of L. Set $G := \langle \Sigma \rangle \leq \mathrm{Aut}(L)$. For calculations in G, we use $(\cdot, \cdot)$ for commutators in their group theoretic meaning. Thus for $g, h \in G$ and $A, B \leq G$, we write $(g, h) := g^{-1}h^{-1}gh$ and $(A, B) =$

$\langle (a, b) \mid a \in A, b \in B \rangle$. Furthermore, we denote the conjugate of A under g by A^g, so $A^g = g^{-1}Ag$. Observe that G preserves the bilinear form f.

DEFINITION 10.2. Let A, B be two Abelian subgroups of G. Following Timmesfeld [15], we call $\langle A, B \rangle$ a *rank 1 group* if the following holds: For each $1 \neq a \in A$ there exists some $1 \neq b \in B$ with $A^b = B^a$, and vice versa. If in addition $a^b = (b^{-1})^a$, then we call the rank 1 group *special*.

THEOREM 10.3. *For $x, y \in \mathcal{E}$ and $s, t \in k$, the group G has the following properties.*

(1) $\exp(y, s)\exp(y, t) = \exp(y, s + t)$. *In particular, U_y is isomorphic to the additive group of k.*

(2) $\exp(y, s)x \in \mathcal{E}$ *with* $(U_x)^{\exp(y,s)} = U_{\exp(y,-s)x}$.

(3) *If* $[x, y] = 0$, *then* $(U_x, U_y) = 1$.

(4) *If* $f(x, y) = 0$, *but* $[x, y] \neq 0$, *then* $(\exp(y, t), \exp(x, s)) = \exp([y, x], ts)$. *In particular, the group $\langle U_x, U_y \rangle$ is nilpotent of class 2 and* $(U_x, U_y) = (u, U_y) = (U_x, v) = U_{[x,y]}$ *for all* $1 \neq u \in U_x$, $1 \neq v \in U_y$.

(5) *If* $f(x, y) = -2$, *then, for* $s \in k$, $s \neq 0$, $\exp(y, -s)\exp(x, s^{-1}t)\exp(y, s) = \exp(x, -s^{-1})\exp(y, -ts)\exp(x, s^{-1})$. *In particular, the group $\langle U_x, U_y \rangle$ is a special rank 1 group in G.*

Proof. All equations can be verified in a straightforward manner by using the definition of $\exp(y, t)$. In the calculations we use the Jacobi identity, associativity of f (e.g., $f(y, [y, z]) = 0$, $f(x, [y, [x, z]]) = 0$ when $f(x, y) = 0$ and $f(x, [y, [x, z]]) = 2f(x, z)$ when $f(x, y) = -2$), and the rewriting rule for $[x, [y, [x, z]]]$ of Lemma 2.2. ∎

Remark 10.4. Theorem 10.3 shows that the set Σ of root groups in G associated to extremal elements of L is a set of so-called *abstract root subgroups* in the sense of Timmesfeld [15]. If Σ is a conjugacy class and G is quasi-simple, then we may apply Timmesfeld's classification [15] of groups generated by abstract root subgroups to determine the structure of L.

Whenever G is an algebraic group its long root subgroups correspond to projective points (that is, one-dimensional linear subspaces) spanned by extremal elements in its Lie algebra $L(G)$. The geometry of these long root subgroups, which is well known, can thus also be studied in the Lie algebra $L(G)$.

For instance, in a Chevalley group, consider the subgroup M generated by two different long root subgroups U_α, U_β, with $\alpha + \beta$ not a root. Then either M is partitioned by the long root subgroups contained in it or it contains no other long root subgroups than U_α and U_β, depending on whether $\alpha - \beta$ is a root or not. In the lemma below, we describe the corresponding behavior in Lie algebras for the case of a partitioning. For recognition of such pairs, we can take the general setup of a Lie algebra containing extremal elements and do not need to assume that it is the Lie algebra of an algebraic group.

LEMMA 10.5. *Let* $x, y \in \mathcal{E}$ *with* $[x, y] = 0$. *Then the following are equivalent*:

(1) *For* $s, t \in k$, $s, t \neq 0$, *the element* $sx + ty$ *is extremal.*

(1′) *There are* $s, t \in k$, $s, t \neq 0$, *such that the element* $sx + ty$ *is extremal.*

(2) $2[y, [x, z]] = f(x, z)y + f(y, z)x$ *for all* $z \in \mathcal{E}$.

(2′) $2[y, [x, z]] = f(x, z)y + f(y, z)x$ *for all* $z \in L$.

Moreover, in these cases, we have $\exp(y, t) \exp(x, s) = \exp(sx + ty, 1)$ *for* $s, t \in k$.

Proof. Note that (2) and (2′) are equivalent by Lemma 2.4. For $z \in \mathcal{E}$, $0 \neq s, t \in k$, we have $[sx + ty, [sx + ty, z]] = s^2 f(x, z)x + t^2 f(y, z)y + 2st[y, [x, z]]$ and $f(sx + ty, z)(sx + ty) = s^2 f(x, z)x + t^2 f(y, z)y + st\big(f(x, z)y + f(y, z)x\big)$. Hence (1′) implies (2). If (2) holds, then $sx + ty$ is extremal and a short calculation shows that $\exp(y, t) \exp(x, s)z = \exp(sx + ty, 1)z$. Note that $[x, [y, z]] = [y, [x, z]]$ here. This yields the result. ∎

COROLLARY 10.6. *If three points on a projective line of* L *represent commuting extremal elements, then the whole line consists of commuting extremal elements.*

An example of the occurrence of projective lines consisting fully of extremal elements as described in the previous lemma is given in the next lemma. Note the correspondence with the group geometries of [15].

LEMMA 10.7. *Let* $x, y \in \mathcal{E}$ *with* $f(x, y) = 0$, *but* $[x, y] \neq 0$. *Then the conditions of Lemma* 10.5 *hold for* x *and* $[x, y]$.

Proof. By Lemma 2.2 we have $2[[x, y], [x, z]] = f(x, [y, z])x + f(x, z)[x, y] + 0$. By the associativity of f (cf. Theorem 2.5) we see that $f(x, [y, z]) = f([x, y], z)$, which proves Condition (2′) of Lemma 10.5, as required. ∎

Many more results in this direction can be derived, such as the nonexistence of a chain x_1, x_2, x_3 of extremal elements such that x_1, x_2 are as x, y in Lemma 10.5, $[x_2, x_3] = 0$, and $f(x_1, x_3) \neq 0$.

To finish, we relate the results on the generation of L by extremal elements and corresponding properties of G. Let $\mathcal{F}(\Sigma)$ denote the graph whose vertices are the elements of Σ and whose edges are the unordered pairs $\{A, B\}$ with $\langle A, B\rangle$ a rank 1 group.

LEMMA 10.8. *Assume that the graph $\mathcal{F}(\Sigma)$ is connected. If $G = \langle U_{x_i} \mid i \in I\rangle$, then L is generated as a Lie algebra by the x_i ($i \in I$).*

Proof. Denote by W the Lie algebra generated by $\{x_i \mid i \in I\}$. Let $z \in \mathcal{E}$ with $f(z, x_i) \neq 0$ for some $i \in I$. Since W is invariant under U_{x_j} for $j \in I$, it is also invariant under G. Hence $[z, x_i] + \frac{1}{2}f(z, x_i)z = \exp(z, 1)\ x_i - x_i \in W$. We apply $\exp(z, 1)$ to this vector and see that $[z, x_i] + \frac{1}{2}f(z, x_i)z + f(z, x_i)z$ is in W. Hence also the difference, which is $f(z, x_i)z$, is in W. Since $f(z, x_i) \neq 0$, we conclude that $z \in W$.

The graph $\mathcal{F}(\Sigma)$ is connected, so we obtain that $\mathcal{E} \subseteq W$; whence $L = W$. ∎

For the case where k is algebraically closed and L is of Chevalley type, we can prove the converse as well.

THEOREM 10.9. *Suppose that k is algebraically closed. Let L be a Lie algebra of Chevalley type over k of characteristic distinct from* 2, 3, *and let G be the corresponding group of automorphisms generated by long root groups. Then $t(L)$, the minimal number of extremal elements generating L, determined in Theorem* 8.2, *is the minimal number of long root groups needed to generate G.*

Proof. Every element of $\mathcal{E}$ is a long root element by Proposition 3.3. Suppose that L is generated by the extremal elements $x_1, \ldots, x_t$. Write $G = \langle U_x \mid x \in \mathcal{E}\rangle$ (as usual) and $H = \langle U_{x_i} \mid i = 1, \ldots, t\rangle$. It suffices to show that H coincides with G. Put $\mathcal{D} = \{x \in \mathcal{E} \mid U_x \subseteq H\}$. Note that, for $x, y \in \mathcal{D}$, also $\exp(x, 1)y \in \mathcal{D}$. Hence, by the argument of the proof of Lemma 2.4 with $\mathcal{D}$ instead of $\mathcal{E}$, we obtain that L is linearly spanned by $\mathcal{D}$.

Subgroups of the algebraic group $\mathrm{GL}(L)$ generated by connected algebraic subgroups are connected algebraic subgroups (see [2, Proposition I.2.2]), so H is a closed algebraic subgroup of the connected linear algebraic group G. Clearly, the derivative $\mathrm{d}\iota$ of the embedding $\iota : H \to G$ at the identity of H is the embedding $L(H) \to L(G)$ of the Lie algebra of H in the Lie algebra of G. As $\mathcal{D}$ linearly spans L, we have $L(H) = L = L(G)$ and so $\mathrm{d}\iota$ is surjective. By Theorem 3.2.21 of [14], this implies that ι is dominant, that is, that H is dense in G. But H is closed as well, so $H = G$. This establishes that G is generated by at most $t(L)$ long root subgroups.

The converse is handled by the previous lemma. ∎

REFERENCES

1. G. Benkart, Inner ideals of Lie algebras, *Trans. Amer. Math. Soc.* **232** (1977), 61–81.
2. A. Borel, "Linear Algebraic Groups," 2nd enlarged ed., Graduate Texts in Mathematics, Vol. 126, Springer-Verlag, NewYork/Berlin, 1991.

3. N. Bourbaki, "Groupes et algèbres de Lie," Chaps. 4–6, Eléménts de Mathématique, Hermann, Paris, 1968.
4. V. I. Chernousov, On the Hasse principle for groups of type E_8, *Dokl. Akad. Nauk SSSR* **306**, No. 25(1989), 1059–1063; *Soviet Math. Dokl.* **39** (1989), 592–596 [Engl. transl.].
5. The GAP Group, GAP–Groups, Algorithms, and Programming, Version 4b5, Aachen, St. Andrews, 1998 [http://www-gap.dcs.st-and.ac.uk/~gap]
6. W. de Graaf, "Lie Algebras: Theory and Algorithm," Elsevier, Amsterdam, 2000.
7. A. I. Kostrikin, "Around Burnside," Springer-Verlag, New York/Berlin, 1990.
8. P. Lalonde and A. Ram, Standard Lyndon bases of Lie algebras and enveloping algebras, *Trans. Amer. Math. Soc.* **347**, No. 5 (1995), 1821–1830.
9. A. M. Cohen, M. van Leeuwen, and B. Lisser, "LiE, a Package for Lie Group Computations," CAN, Amsterdam, 1992 [http://www.win.tue.nl/math/dw/LiEform/form.html].
10. A. Premet, Lie algebras without strong degeneration, *Math. USSR Sbo.* **57** (1987), 151–164.
11. A. Premet and H. Strade, Simple Lie algebras of small characteristic. I. Sandwich elements, *J. Algebra* **189** (1997), 419–480.
12. A. A. Premet and I. D. Suprunenko, Quadratic modules for Chevalley groups of fields of odd characteristics, *Math. Nachr.* **110** (1983), 65–96.
13. C. Reutenauer, "Free Lie Algebras," London Math. Society Monographs (New Series), Vol. 7, Oxford Univ. Press, Oxford, UK, 1993.
14. T. A. Springer, "Linear Algebraic Groups," Birkhäuser, Basel, 1984.
15. F. G. Timmesfeld, Groups generated by abstract root groups and quadratic action, *Adv. in Math.* **142** (1999), 1–150.
16. M. Vaughan-Lee and E. I. Zel'manov, Upper bounds in the restricted Burnside problem, *J. Algebra* **162** (1993), 107–145.
17. E. I. Zel'manov, Absolute zero divisors in Jordan pairs and Lie algebras, *Math. USSR Sbo.* **40** (1981), 549–565.
18. E. I. Zel'manov and A. I. Kostrikin, A theorem on sandwich algebras, *in* "Proceedings of the Steklov Institute of Mathematics, 1991," Issue 4.