Introduction to Reinforcement Learning and multi-armed bandits

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Outline of the course

- Part 1: Introduction to Reinforcement Learning and Dynamic Programming
  - Dynamic programming: value iteration, policy iteration
  - Q-learning.
- Part 2: Approximate DP and RL
  - L_∞-norm performance bounds
  - Sample-based algorithms.
  - Links with statistical learning
- Part 3: Intro to multi-armed bandits
  - The stochastic bandit: UCB
  - The adversarial bandit: EXP3
  - Approximation of Nash equilibrium
  - Monte-Carlo Tree Search

Part 1: Introduction to Reinforcement Learning and Dynamic Programming

A few general references:

- *Neuro Dynamic Programming*, Bertsekas et Tsitsiklis, 1996.
- *Introduction to Reinforcement Learning*, Sutton and Barto, 1998.
- *Markov Decision Problems*, Puterman, 1994.
- *Algorithms for Reinforcement Learning*, Szepesvári, 2009.

Introduction to Reinforcement Learning (RL)

- Learn to make good decisions in unknown environments
- Learning from experience: success or failures
- Examples: learning to ride a bicycle, play chess, autonomous robotics, operation research, playing in stochastic market, ...

I learned to ride with RL...

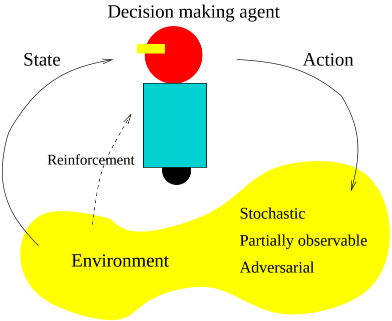
A few applications

- TD-Gammon. [Tesauro 1992-1995]: Backgammon.
- KnightCap [Baxter et al. 1998]: chess ($\simeq$2500 ELO)
- Robotics: juggling, acrobots [Schaal and Atkeson, 1994]
- Mobile robot navigation [Thrun et al., 1999]
- Elevator controller [Crites et Barto, 1996],
- Packet Routing [Boyan et Littman, 1993],
- Job-Shop Scheduling [Zhang et Dietterich, 1995],
- Production manufacturing optimization[Mahadevan et al., 1998],
- Game of poker (Bandit algo for Nash computation)
- Game of go (hierarchical bandits, UCT)

`http://www.ualberta.ca/∼szepesva/RESEARCH/RLApplications.html`


INFINITE MARIO BROS
(NOT Sequential Learning)
http://www

Reinforcement Learning

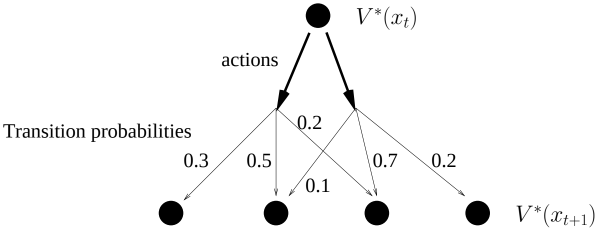


- **Environment:** can be stochastic (Tetris), adversarial (Chess), partially unknown (bicycle), partially observable (robot)
- **Available information:** the reinforcement (may be delayed)
- **Goal:** maximize the expected sum of future rewards.

Problem: How to sacrify a short term small reward to priviledge larger rewards in the long term?

Optimal value function

- Gives an evaluation of each state if the agent plays optimally.
- Ex: in a stochastic environment:



- Bellman equation:
  $V^*(x_t) = \max_{a \in A} \left[r(x_t, a) + \sum_y p(y|x_t, a) V^*(y) \right]$
- Temporal difference: $\delta_t = V^*(x_{t+1}) + r(x_t, a_t) - V^*(x_t)$
- If V^* is known, then when choosing the optimal action a_t, $\mathbb{E}[\delta_t] = 0$ (i.e., in average there is no surprise)

Challenges of RL

- Environment may be stochastic, adversarial, partially observable...
- The state-dynamics and reward functions are unknown: we need to combine
  - Learning
  - Planning
- The curse of dimensionality: We need to rely on *approximations* for representing the value function and the optimal policy.

Introduction to Dynamic Programming

A **Markov Decision Process** (X, A, p, r) defines a discrete-time process $(x_t) \in X$ where:

- X: **state space**
- A: **action space** (or decisions)
- **State dynamics**: All relevant information about future is included in the current state and action (Markov property)

  $$\mathbb{P}(x_{t+1} \mid x_t, x_{t-1}, \ldots, x_0, a_t, a_{t-1}, \ldots, a_0) = \mathbb{P}(x_{t+1} \mid x_t, a_t)$$

  Thus we define the **transition probabilities** $p(y|x, a)$
- **Reinforcement** (or **reward**): $r(x, a)$ is obtained when choosing action a in state x.

Definition of policy

Policy $\pi = (\pi_1, \pi_2, \dots)$, where at time t,

$$\pi_t : X \to A$$

maps an action $\pi_t(x)$ to any possible state x.

Given a policy π the process $(x_t)_{t \geq 0}$ is a Markov chain with transition probabilities

$$p(x_{t+1}|x_t) = p(x_{t+1}|x_t, \pi_t(x_t)).$$

When the policy is independent of time, $\pi = (\pi, \pi, \dots, \pi)$, the policy is called *stationary* (or *Markovian*).

Performance of a policy

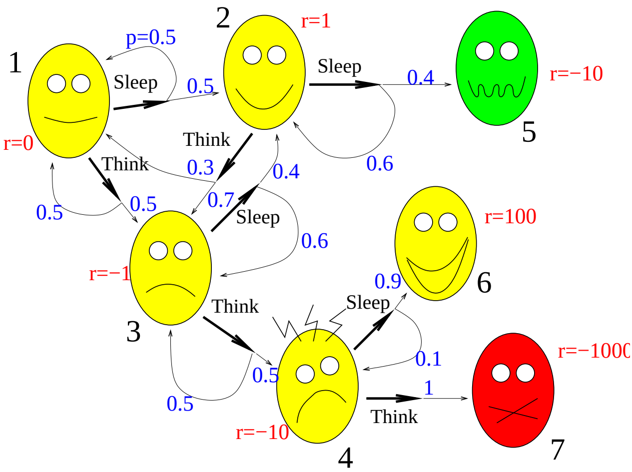
For any policy π, define the **value function** V^π:

Infinite horizon:

- Discounted: $V^\pi(x) = \mathbb{E}\Big[\sum_{t=0}^{\infty} \gamma^t r(x_t, a_t) \,|\, x_0 = x; \pi\Big]$,
  where $0 \leq \gamma < 1$ is the discount factor
- Undiscounted: $V^\pi(x) = \mathbb{E}\Big[\sum_{t=0}^{\infty} r(x_t, a_t) \,|\, x_0 = x; \pi\Big]$
- Average: $V^\pi(x) = \lim_{T\to\infty} \frac{1}{T}\mathbb{E}\Big[\sum_{t=0}^{T-1} r(x_t, a_t) \,|\, x_0 = x; \pi\Big]$

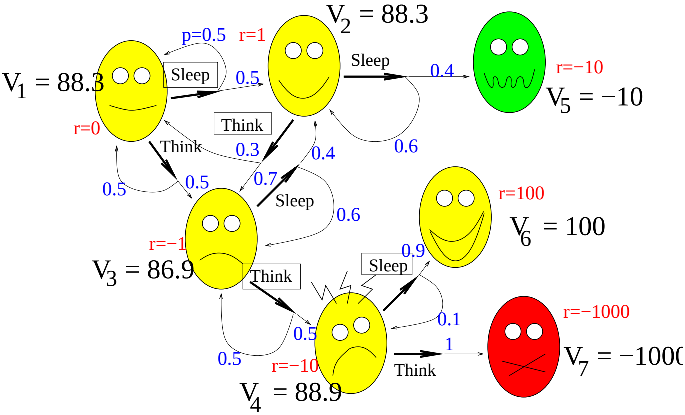
Finite horizon: $V^\pi(x, t) = \mathbb{E}\Big[\sum_{s=t}^{T-1} r(x_s, a_s) + R(x_T) \,|\, x_t = x; \pi\Big]$

The dilemma of the Netadis SS student



You try to maximize the sum of rewards!

Solution of the Netadis SS student



$V_5 = -10,\ V_6 = 100,\ V_7 = -1000,$
$V_4 = -10 + 0.9V_6 + 0.1V_4 \simeq 88.9.$
$V_3 = -1 + 0.5V_4 + 0.5V_3 \simeq 86.9.$ $V_2 = 1 + 0.7V_3 + 0.3V_1$ and
$V_1 = \max\{0.5V_2 + 0.5V_1, 0.5V_3 + 0.5V_1\}$, thus: $V_1 = V_2 = 88.3.$

Infinite horizon, discounted problems

For any stationary policy π, define the **value function** V^π as:

$$V^\pi(x) = \mathbb{E}\Big[\sum_{t=0}^{\infty} \gamma^t r(x_t, \pi(x_t)) \,\Big|\, x_0 = x; \pi\Big],$$

where $0 \leq \gamma < 1$ a discount factor (which relates rewards in the future compared to current rewards).

Bellman equation for V^π

Proposition 1 (Bellman equation).

For any policy π, V^π satisfies:

$$V^\pi(x) = r(x, \pi(x)) + \gamma \sum_{y \in X} p(y|x, \pi(x)) V^\pi(y),$$

Thus V^π is the fixed point of the **Bellman operator** $\mathcal{T}^\pi$ (i.e., $V^\pi = \mathcal{T}^\pi V^\pi$) where $\mathcal{T}^\pi W$ is defined as

$$\mathcal{T}^\pi W(x) = r(x, \pi(x)) + \gamma \sum_{y} p(y|x, \pi(x)) W(y)$$

Using matrix notations, $\mathcal{T}^\pi W = r^\pi + \gamma P^\pi W$, where $r^\pi(x) = r(x, \pi(x))$ and $P^\pi(x, y) = p(y|x, \pi(x))$.

Proof of Proposition 1

$$
\begin{aligned}
V^\pi(x) &= \mathbb{E}\Big[\sum_{t\geq 0} \gamma^t r(x_t, \pi(x_t)) \,|\, x_0 = x; \pi\Big] \\
&= r(x, \pi(x)) + \mathbb{E}\Big[\sum_{t\geq 1} \gamma^t r(x_t, \pi(x_t)) \,|\, x_0 = x; \pi\Big] \\
&= r(x, \pi(x)) + \gamma \sum_y P(x_1 = y \,|\, x_0 = x; \pi) \\
&\qquad \mathbb{E}\Big[\sum_{t\geq 1} \gamma^{t-1} r(x_t, \pi(x_t)) \,|\, x_1 = y; \pi\Big] \\
&= r(x, \pi(x)) + \gamma \sum_y p(y|x, \pi(x)) V^\pi(y).
\end{aligned}
$$

Bellman equation for V^*

Define the **optimal value function**: $V^* = \sup_\pi V^\pi$.

Proposition 2 (Dynamic programming equation).

V^* satisfies:

$$V^*(x) = \max_{a \in A} \Big[r(x,a) + \gamma \sum_{y \in X} p(y|x,a) V^*(y) \Big].$$

Thus V^* is the fixed point of the **Dynamic programming operator** $\mathcal{T}$ (i.e., $V^* = \mathcal{T} V^*$) where $\mathcal{T} W$ is defined as

$$\mathcal{T} W(x) = \max_{a \in A} \Big[r(x,a) + \gamma \sum_{y \in X} p(y|x,a) W(y) \Big].$$

Proof of Proposition 2

And for all policy $\pi = (a, \pi')$ (not necessarily stationary),

$$\begin{aligned}
V^*(x) &= \max_\pi \mathbb{E}\Big[\sum_{t\geq 0} \gamma^t r(x_t, \pi(x_t)) \,|\, x_0 = x; \pi\Big] \\
&= \max_{(a,\pi')} \Big[r(x,a) + \gamma \sum_y p(y|x,a) V^{\pi'}(y)\Big] \\
&= \max_a \Big[r(x,a) + \gamma \sum_y p(y|x,a) \max_{\pi'} V^{\pi'}(y)\Big] \qquad (1) \\
&= \max_a \Big[r(x,a) + \gamma \sum_y p(y|x,a) V^*(y)\Big].
\end{aligned}$$

where (1) holds since:

- $\max_{\pi'} \sum_y p(y|x,a) V^{\pi'}(y) \leq \sum_y p(y|x,a) \max_{\pi'} V^{\pi'}(y)$
- Let $\bar{\pi}$ be the policy defined by $\bar{\pi}(y) = \arg\max_{\pi'} V^{\pi'}(y)$. Thus $\sum_y p(y|x,a) \max_{\pi'} V^{\pi'}(y) = \sum_y p(y|x,a) V^{\bar{\pi}}(y) \leq \max_{\pi'} \sum_y p(y|x,a) V^{\pi'}(y)$.

Properties of the Bellman operators

- **Monotonicity**: If $W_1 \leq W_2$ (componentwise) then

$$\mathcal{T}^\pi W_1 \leq \mathcal{T}^\pi W_2, \text{ and } \mathcal{T} W_1 \leq \mathcal{T} W_2.$$

- **Contraction in max-norm**: For any vectors W_1 and W_2,

$$\begin{aligned} ||\mathcal{T}^\pi W_1 - \mathcal{T}^\pi W_2||_\infty &\leq \gamma ||W_1 - W_2||_\infty, \\ ||\mathcal{T} W_1 - \mathcal{T} W_2||_\infty &\leq \gamma ||W_1 - W_2||_\infty. \end{aligned}$$

Indeed, for all $x \in X$,

$$\begin{aligned} |\mathcal{T} W_1(x) - \mathcal{T} W_2(x)| &= \Big| \max_a \Big[r(x,a) + \gamma \sum_y p(y|x,a) W_1(y) \Big] \\ &\quad - \max_a \Big[r(x,a) + \gamma \sum_y p(y|x,a) W_2(y) \Big] \Big| \\ &\leq \gamma \max_a \sum_y p(y|x,a) |W_1(y) - W_2(y)| \\ &\leq \gamma ||W_1 - W_2||_\infty \end{aligned}$$

Properties of the value functions

Proposition 3.

1. *V^π is the unique fixed-point of $\mathcal{T}^\pi$*

$$V^\pi = \mathcal{T}^\pi V^\pi.$$

2. *V^* is the unique fixed-point of $\mathcal{T}$:*

$$V^* = \mathcal{T} V^*.$$

3. *For any policy π, we have $V^\pi = (I - \gamma P^\pi)^{-1} r^\pi$*
4. *The policy defined by*

$$\pi^*(x) \in \arg\max_{a \in A} \left[r(x,a) + \gamma \sum_y p(y|x,a) V^*(y) \right]$$

is optimal (and stationary)

Proof of Proposition 3

1. From Proposition 1, V^π is a fixed point of $\mathcal{T}^\pi$. Uniqueness comes from the contraction property of $\mathcal{T}^\pi$.
2. Idem for V^*.
3. $V^\pi = \mathcal{T}^\pi V^\pi = r^\pi + \gamma P^\pi V^\pi$. Thus $(I - \gamma P^\pi)V^\pi = r^\pi$. Now P^π is a stochastic matrix (whose eingenvalues have a modulus ≤ 1), thus the eing. of $(I - \gamma P^\pi)$ have a modulus $\geq 1 - \gamma > 0$, thus is invertible.
4. From the definition of π^*, we have

$$\mathcal{T}^{\pi^*} V^* = \mathcal{T} V^* = V^*$$

Thus V^* is the fixed-point of $\mathcal{T}^{\pi^*}$. But, by definition, V^{π^*} is the fixed-point of $\mathcal{T}^{\pi^*}$ and since there is uniqueness of the fixed-point, $V^{\pi^*} = V^*$ and π^* is optimal.

Value Iteration

Proposition 4.

- *For any bounded* π *and* V_0*, define* $V_{k+1} = \mathcal{T}^\pi V_k$*. Then* $V_k \to V^\pi$*.*
- *For any bounded* V_0*, define* $V_{k+1} = \mathcal{T} V_k$*. Then* $V_k \to V^*$*.*

Proof.

$$||V_{k+1} - V^*|| = ||\mathcal{T} V_k - \mathcal{T} V^*|| \leq \gamma ||V_k - V^*|| \leq \gamma^{k+1} ||V_0 - V^*|| \to 0$$

(idem for V^π) $\square$

Variant: asynchronous iterations

Policy Iteration

Choose any initial policy π_0. Iterate:

1. **Policy evaluation**: compute V^{π_k}.
2. **Policy improvement**: π_{k+1} greedy w.r.t. V^{π_k}:

$$\pi_{k+1}(x) \in \arg\max_{a \in A} \Big[r(x,a) + \gamma \sum_y p(y|x,a) V^{\pi_k}(y) \Big],$$

(i.e. $\pi_{k+1} \in \arg\max_\pi \mathcal{T}^\pi V^{\pi_k}$)

Stop when $V^{\pi_k} = V^{\pi_{k+1}}$.

Proposition 5.

Policy iteration generates a sequence of policies with increasing performance ($V^{\pi_{k+1}} \geq V^{\pi_k}$) and (in the case of finite state and action spaces) terminates in a finite number of steps with the optimal policy π^*.

Proof of Proposition 5

From the definition of the operators $\mathcal{T}$, $\mathcal{T}^{\pi_k}$, $\mathcal{T}^{\pi_{k+1}}$ and from π_{k+1},

$$V^{\pi_k} = \mathcal{T}^{\pi_k} V^{\pi_k} \leq \mathcal{T} V^{\pi_k} = \mathcal{T}^{\pi_{k+1}} V^{\pi_k}, \tag{2}$$

and from the monotonicity of $\mathcal{T}^{\pi_{k+1}}$, we have

$$V^{\pi_k} \leq \lim_{n\to\infty} (\mathcal{T}^{\pi_{k+1}})^n V^{\pi_k} = V^{\pi_{k+1}}.$$

Thus $(V^{\pi_k})_k$ is a non-decreasing sequence. Since there is a finite number of possible policies (finite state and action spaces), the stopping criterion holds for a finite k; We thus have equality in (2), thus

$$V^{\pi_k} = \mathcal{T} V^{\pi_k}$$

so $V^{\pi_k} = V^*$ and π_k is an optimal policy.

Back to Reinforcement Learning

What if the transition probabilities $p(y|x, a)$ and the reward functions $r(x, a)$ are unknown?
In DP, we used their knowledge

- in value iteration:

$$V_{k+1}(x) = \mathcal{T} V_k(x) = \max_a \left[r(x, a) + \gamma \sum_y p(y|x, a) V_k(y) \right].$$

- in policy iteration:
  - when computing V^{π_k} (which requires iterating $\mathcal{T}^{\pi_k}$)
  - when computing the greedy policy:

$$\pi_{k+1}(x) \in \arg\max_{a \in A} \left[r(x, a) + \gamma \sum_y p(y|x, a) V^{\pi_k}(y) \right],$$

RL = introduction of 2 ideas: **Q-functions** and **sampling**.

Definition of the Q-value function

Define the Q-value function $Q^\pi : X \times A \to \mathbb{R}$: for a policy π,

$$Q^\pi(x,a) = \mathbb{E}\big[\sum_{t\geq 0} \gamma^t r(x_t, a_t) | x_0 = x, a_0 = a, a_t = \pi(x_t), t \geq 1\big]$$

and the optimal Q-value function $Q^*(x,a) = \max_\pi Q^\pi(x,a)$.

Proposition 6.

Q^π and Q^* satisfy the Bellman equations:

$$\begin{aligned} Q^\pi(x,a) &= r(x,a) + \gamma \sum_{y \in X} p(y|x,a) Q^\pi(y, \pi(y)) \\ Q^*(x,a) &= r(x,a) + \gamma \sum_{y \in X} p(y|x,a) \max_{b \in A} Q^\pi(y,b) \end{aligned}$$

Idea: compute Q^* and then $\pi^*(x) \in \arg\max_a Q^*(x,a)$.

Q-learning algorithm [Watkins, 1989]

Builds a sequence of Q-value functions Q_k.
Whenever a transition $x_t, a_t \xrightarrow{r_t} x_{t+1}$ occurs, update the Q-value:

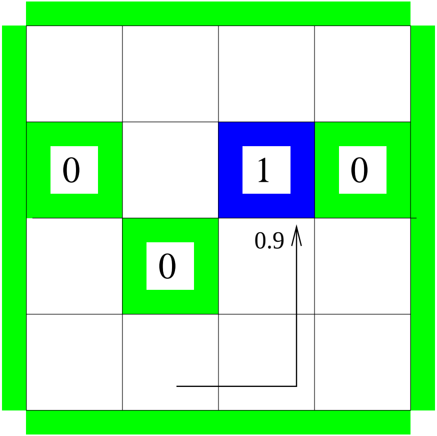
$$Q_{k+1}(x_t, a_t) = Q_k(x_t, a_t) + \eta_k(x_t, a_t)\underbrace{\Big[r_t + \gamma \max_{b \in A} Q_k(x_{t+1}, b) - Q_k(x_t, a_t) \Big]}_{\text{temporal difference}}.$$

Proposition 7 (Watkins et Dayan, 1992).
Assume that all state-action pairs (x, a) are visited infinitely often and that the learning steps satisfy for all x, a, $\sum_{k\geq 0} \eta_k(x, a) = \infty$, $\sum_{k\geq 0} \eta_k^2(x, a) < \infty$, *then* $Q_k \xrightarrow{a.s.} Q^*$.

The proof relies on Stochastic Approximation for estimating the fixed-point of a contraction mapping.

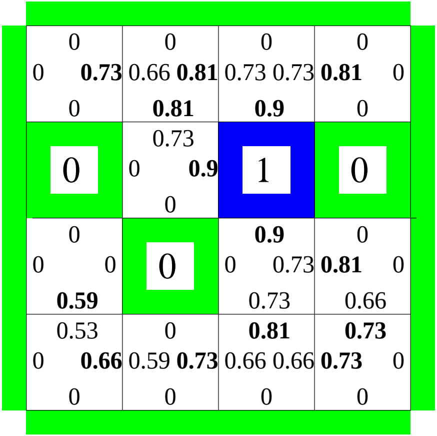
Q-learning algorithm

Deterministic case, discount factor $\gamma = 0.9$. Take steps $\eta = 1$.



After transition $x, a \xrightarrow{r} y$ update $Q_{k+1}(x, a) = r + \gamma \max_{b \in A} Q_k(y, b)$

Optimal Q-values



Bellman's equation: $Q^*(x, a) = \gamma \max_{b \in A} Q^*(\text{next-state}(x, a), b).$

First conclusions

When the state-space is finite and "small":

- If transition probabilities and rewards are known, then DP algorithms (value iteration, policy iteration) compute the optimal solution
- Otherwise, use sampling techniques and RL algorithms (Q-learning, TD(λ)) apply

2 main issues:

- Usually state-space is large (infinite)! We need to build **approximate solutions**.
- We need to design clever **exploration strategies**.