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Sensitivity analysis using Itô-Malliavin calculus and martingales. Numerical Implementation.

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SENSITIVITY ANALYSIS USING ITÔ-MALLIAVIN CALCULUS AND MARTINGALES. NUMERICAL IMPLEMENTATION.

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Abstract. This note is a guide to numerical implementation of several methods introduced in the companion paper Gobet and Munos, *Sensitivity Analysis using Itô-Malliavin Calculus and Martingales, Application to Stochastic Optimal Control*, namely the computation of the sensitivity of a cost function with respect to parameters of the process dynamics. Four methods are described based on *Path-wise*, *Malliavin calculus*, *adjoint* and *martingale* approaches.

Key words. Numerical implementation, Sensitivity analysis, Malliavin calculus.

AMS subject classifications. 90C31, 93E20, 60H30

1. Introduction. We refer the reader to [GM02] for all notations, hypotheses, and theoretical results. Consider the stochastic differential equation

$$X_t = x + \int_0^t b(s, X_s, \alpha)\, ds + \sum_{j=1}^{q} \int_0^t \sigma_j(s, X_s, \alpha)\, dW_s^j \tag{1.1}$$

where $X_t \in \mathbb{R}^d$, α is a parameter (taking values in $\mathcal{A} \subset \mathbb{R}^m$) and $(W_t)_{0 \le t \le T}$ is a standard Brownian motion in $\mathbb{R}^q$. In this note, we restrict the study to the case $q = d$ and σ being invertible.

Our goal is to evaluate the sensitivity w.r.t. α of the cost function

$$J(\alpha) = \mathbb{E}\left(\int_0^T f(X_t)dt + g(X_T)\right), \tag{1.2}$$

which depends on instantaneous and terminal costs f and g.

We recall the notation for differentiation. The derivative w.r.t. α is denoted with a dot, for example $\dot{X}_t = \nabla_\alpha X_t = (\partial_{\alpha_1} X_t, \dots, \partial_{\alpha_m} X_t) = (\dot{X}_{1,t}, \dots, \dot{X}_{m,t})$ is considered as a $d \times m$ matrix. The derivative w.r.t. the state (i.e. the gradient) is denoted with a prime, for example, $b'_s = \nabla_x b_s = (\partial_{x_1} b_s, \dots, \partial_{x_d} b_s) = (\partial_{x_1} b(s, X_s, \alpha), \dots, \partial_{x_d} b(s, X_s, \alpha))$.

2. Path-wise approach. To the diffusion X, we associate the path-wise derivative of X_t with respect to α, which we denote $\dot{X}_t$. This process solves

$$\dot{X}_t = \int_0^t \left(\dot{b}_s + b'_s\, \dot{X}_s\right) ds + \sum_{j=1}^{d} \int_0^t \left(\dot{\sigma}_{j,s} + \sigma'_{j,s}\, \dot{X}_s\right) dW_s^j. \tag{2.1}$$

Proposition 1.1 in [GM02] provides the sensitivity of J w.r.t. α using the path-wise approach:

$$\dot{J}(\alpha) = \mathbb{E}\Big(\int_0^T f'(X_t)\dot{X}_t dt + g'(X_T)\dot{X}_T\Big). \tag{2.2}$$

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3. Malliavin calculus approach in the elliptic case. To the diffusion X, we associate its flow, i.e. the Jacobian matrix $Y_t := \nabla_x X_t$ and the inverse of its flow $Z_t = Y_t^{-1}$. These processes solve

$$Y_t = \mathrm{I}_d + \int_0^t b'_s \, Y_s \, ds + \sum_{j=1}^d \int_0^t \sigma'_{j,s} \, Y_s \, dW_s^j, \tag{3.1}$$

$$Z_t = \mathrm{I}_d - \int_0^t Z_s (b'_s - \sum_{j=1}^d (\sigma'_{j,s})^2) \, ds - \sum_{j=1}^d \int_0^t Z_s \sigma'_{j,s} \, dW_s^j. \tag{3.2}$$

Then, from Proposition 2.5, one has

$$\partial_{\alpha_k} J(\alpha) = \mathbb{E}\Big(\int_0^T f(X_t) H_{k,t} dt + g(X_T) H_{k,T} \Big)$$

with

$$H_{k,t} = \frac{1}{t} \delta\big([\sigma_\cdot^{-1} \, Y_\cdot \, Z_t \, \dot{X}_{k,t}]^* \big)$$

where $\dot{X}_{k,t} = \partial_{\alpha_k} X_t$ is the k^{th} column of $\dot{X}_t$.

Let us write $u_s = \sigma_s^{-1} Y_s$ and $F_{k,t} = Z_t \dot{X}_{k,t}$. Call $u_{i,s}$ the i^{th} column of u_s (thus $u_{i,s} = \sigma_s^{-1} Y_{i,s}$ where $Y_{i,s}$ stands for the i^{th} column of Y_s). Thus $u_s F_{k,t} = \sum_i u_{i,s} F_{i,k,t}$ where $F_{i,k,t}$ is the i^{th} component of $F_{k,t}$. Hence,

$$H_{k,t} = \frac{1}{t} \sum_{i=1}^d \delta(u_{i,\cdot}^* F_{i,k,t}) = \frac{1}{t} \sum_{i=1}^d [\delta(u_{i,\cdot}^*) F_{i,k,t} - \int_0^t \mathcal{D}_s F_{i,k,t} u_{i,s} ds]. \tag{3.3}$$

Since u is a square integrable adapted process, we have

$$\delta(u_{i,\cdot}^*) = \int_0^t u_{i,s}^* dW_s$$

thus

$$\frac{1}{t} \sum_{i=1}^d \delta(u_{i,\cdot}^*) F_{i,k,t} = \frac{1}{t} \sum_{i=1}^d F_{i,k,t} \int_0^t Y_{i,s}^* (\sigma_s^{-1})^* dW_s. \tag{3.4}$$

3.1. Computation of $\mathcal{D}F$. In this paragraph, t is fixed and for simplicity we omit to indicate this dependency.

We make use of the following result: let A be a $p_1 \times p_2$ matrix and b a vector of size p_2, then

$$\mathcal{D}(Ab) = \sum_{k=1}^{p_2} \mathcal{D}(A_k) b_k + A \mathcal{D} b$$

where A_k is the k^{th} column of A.

Since

$$\mathcal{D}F_k = \mathcal{D}(Z \dot{X}_k) = \sum_{j=1}^d \mathcal{D}(Z_j) \dot{X}_{jk} + Z \mathcal{D}(\dot{X}_k), \tag{3.5}$$

where $\dot{X}_{jk} = \partial_{\alpha_k} X_j$ is the $(j,k)^{th}$ element of matrix $\dot{X}$ and Z_j is the j^{th} column of Z, we need to compute $\mathcal{D}Z$ and $\mathcal{D}\dot{X}$.

Since $YZ = I$, we have $YZ_j = e^j$ where e^j is a vector with 0s except for the j^{th} component which is 1. From the above result,

$$\mathcal{D}(YZ_j) = \sum_{i=1}^{d} \mathcal{D}(Y_i) Z_{ij} + Y\mathcal{D}(Z_j) = 0$$

hence

$$\mathcal{D}(Z_j) = -Z \sum_{i=1}^{d} \mathcal{D}(Y_i) Z_{ij}$$

which, replaced in (3.5), gives

$$\begin{aligned} \mathcal{D}F_k &= -Z \sum_{i,j=1}^{d} \mathcal{D}(Y_i) Z_{ij} \dot{X}_{jk} + Z\mathcal{D}(\dot{X}_k) \\ &= Z\big(-\sum_{i=1}^{d} \mathcal{D}(Y_i)(Z\dot{X}_k)_i + \mathcal{D}(\dot{X}_k)\big). \end{aligned} \tag{3.6}$$

We need to compute the Malliavin derivative of Y and $\dot{X}$. We know that the Malliavin derivative of X is

$$\mathcal{D}_s X_t = Y_t Z_s \sigma_s \mathbf{1}_{s \le t}.$$

We now describe how to compute the Malliavin derivative of a process relative to X and apply this to compute $\mathcal{D}Y$ and $\mathcal{D}\dot{X}$.

3.2. Malliavin derivative of a process related to X. Let $\hat{X} \in \mathbb{R}^d$ be a process satisfying

$$\hat{X}_t = \hat{x} + \int_0^t \hat{b}(s, X_s, \hat{X}_s) ds + \sum_{j=1}^{d} \int_0^t \hat{\sigma}_j(s, X_s, \hat{X}_s) dW_s^j \tag{3.7}$$

For simplicity we write $\hat{b}_s = \hat{b}(s, X_s, \hat{X}_s)$ and $\hat{\sigma}_s = \hat{\sigma}(s, X_s, \hat{X}_s)$.

Call $\hat{Y}_t = \nabla_{\hat{x}} \hat{X}_t$ its Jacobian and $\hat{Z}_t = \hat{Y}_t^{-1}$ its inverse, which solve

$$\hat{Y}_t = I_d + \int_0^t \nabla_{\hat{x}} \hat{b}_s \hat{Y}_s ds + \sum_{j=1}^{d} \int_0^t \nabla_{\hat{x}} \hat{\sigma}_{j,s} \hat{Y}_s dW_s^j, \tag{3.8}$$

$$\hat{Z}_t = I_d - \int_0^t \hat{Z}_s \big(\nabla_{\hat{x}} \hat{b}_s - \sum_{j=1}^{d} (\nabla_{\hat{x}} \hat{\sigma}_{j,s})^2\big) ds - \int_0^t \hat{Z}_s \sum_{j=1}^{d} \nabla_{\hat{x}} \hat{\sigma}_{j,s} dW_s^j. \tag{3.9}$$

Consider the column vector (of size $2d$) $\begin{pmatrix} X_t \\ \hat{X}_t \end{pmatrix}$, which solves the system

$$\begin{pmatrix} X_t \\ \hat{X}_t \end{pmatrix} = \begin{pmatrix} x \\ \hat{x} \end{pmatrix} + \int_0^t \begin{pmatrix} b_s \\ \hat{b}_s \end{pmatrix} ds + \sum_{j=1}^{d} \int_0^t \begin{pmatrix} \sigma_s \\ \hat{\sigma}_s \end{pmatrix} dW_s^j.$$

Then, its Jacobian $\begin{pmatrix} \nabla_x X_t & \nabla_{\hat{x}} X_t \\ \nabla_x \hat{X}_t & \nabla_{\hat{x}} \hat{X}_t \end{pmatrix} = \begin{pmatrix} Y_t & 0 \\ V_t & \hat{Y}_t \end{pmatrix}$ with $V_t := \nabla_x \hat{X}_t$ solves

$$\begin{pmatrix} Y_t & 0 \\ V_t & \hat{Y}_t \end{pmatrix} = I_{2d} + \int_0^t \begin{pmatrix} \nabla_x b_s & 0 \\ \nabla_x \hat{b}_s & \nabla_{\hat{x}} \hat{b}_s \end{pmatrix} \begin{pmatrix} Y_s & 0 \\ V_s & \hat{Y}_s \end{pmatrix} ds$$
$$+ \sum_{j=1}^d \int_0^t \begin{pmatrix} \nabla_x \sigma_{j,s} & 0 \\ \nabla_x \hat{\sigma}_{j,s} & \nabla_{\hat{x}} \hat{\sigma}_{j,s} \end{pmatrix} \begin{pmatrix} Y_s & 0 \\ V_s & \hat{Y}_s \end{pmatrix} dW_s^j.$$

This system is equivalent to the SDEs (3.1), (3.8) and

$$V_t = \int_0^t (\nabla_x \hat{b}_s Y_s + \nabla_{\hat{x}} \hat{b}_s V_s) ds + \sum_{j=1}^d \int_0^t (\nabla_x \hat{\sigma}_s Y_s + \nabla_{\hat{x}} \hat{\sigma}_s V_s) dW_s^j. \tag{3.10}$$

Note also that the inverse of the Jacobian above $\begin{pmatrix} Y_t & 0 \\ V_t & \hat{Y}_t \end{pmatrix}$ is $\begin{pmatrix} Z_t & 0 \\ -\hat{Z}_t V_t Z_t & \hat{Z}_t \end{pmatrix}$. Thus, the Malliavin derivative of $\begin{pmatrix} X_t \\ \hat{X}_t \end{pmatrix}$ is

$$\mathcal{D}_s \begin{pmatrix} X_t \\ \hat{X}_t \end{pmatrix} = \begin{pmatrix} Y_t & 0 \\ V_t & \hat{Y}_t \end{pmatrix} \begin{pmatrix} Z_s & 0 \\ -\hat{Z}_s V_s Z_s & \hat{Z}_s \end{pmatrix} \begin{pmatrix} \sigma_s \\ \hat{\sigma}_s \end{pmatrix} \mathbf{1}_{s \le t}$$

from which we deduce the Malliavin derivative of $\hat{X}_t$:

$$\mathcal{D}_s(\hat{X}_t) = \left[(V_t - \hat{Y}_t \hat{Z}_s V_s) Z_s \sigma_s + \hat{Y}_t \hat{Z}_s \hat{\sigma}_s \right] \mathbf{1}_{s \le t}. \tag{3.11}$$

In order to compute $\mathcal{D}_s(\hat{X}_t)$ we thus need to solve the system (1.1), (3.1), (3.2), (3.7), (3.8), (3.9), and (3.10).

3.3. Malliavin derivative of Y. Now, consider $Y_{i,t}$ the i^{th} column of Y_t and apply the results of previous section to $\hat{X}_t = Y_{i,t}$. From (3.1) we have $\hat{b}_s = b'_s \hat{X}_s$ and $\hat{\sigma}_{j,s} = \sigma'_{j,s} \hat{X}_s$. We observe that $\nabla_{\hat{x}} \hat{b}_s = b'_s$ and $\nabla_{\hat{x}} \hat{\sigma}_{j,s} = \sigma'_{j,s}$, thus the dynamics on $\hat{Y}$ and $\hat{Z}$ are the same as those on Y and Z respectively, and since their initial conditions are the same ($Y_0 = \hat{Y}_0 = I_d$ and $Z_0 = \hat{Z}_0 = I_d$), we have for all s, $\hat{Y}_s = Y_s$ and $\hat{Z}_s = Z_s$. Thus, in order to compute the Malliavin derivative of $Y_{i,t}$, we just need to introduce the process $V_t^i = \nabla_x Y_{i,t}$ and solve (1.1), (3.1), (3.2) and (3.10), which reads (noticing that $\nabla_x \hat{b} = \nabla_x (b' \hat{X}) = \sum_l Y_{li} \nabla_x b'_l$ where $b'_l = \partial_{x_l} b$ and Y_{li} is the $(l,i)^{th}$ element of Y, and $\nabla_x \hat{\sigma}_j = \sum_l Y_{li} \nabla_x \sigma'_{l,j}$, where $\sigma'_{l,j} = \partial_{x_l} \sigma_j$):

$$V_t^i = \int_0^t [(\sum_{l=1}^d Y_{li,s} \nabla_x b'_{l,s}) Y_s + b'_s V_s^i] ds + \sum_{j=1}^d \int_0^t [(\sum_{l=1}^d Y_{li,s} \nabla_x \sigma'_{l,j,s}) Y_s + \sigma'_{j,s} V_s^i] dW_s^j. \tag{3.12}$$

Then we deduce from (3.11) that the Malliavin derivative of Y_i is

$$\mathcal{D}_s(Y_{i,t}) = \left[(V_t^i - Y_t Z_s V_s^i) Z_s \sigma_s + Y_t Z_s \sigma_s^i \right] \mathbf{1}_{s \le t} \tag{3.13}$$

with σ_s^i the matrix whose j^{th} columns are $\sigma'_{j,s} Y_{i,s}$.

3.4. Malliavin derivative of $\dot{X}$. Since $\dot{X}_t$ solves equation (2.1), we apply the results of section 3.2 with $\hat{X}_t = \dot{X}_{k,t}$, the k^{th} column of $\dot{X}_t$ (its derivative w.r.t. to the parameter α_k). Thus $\hat{b}_s = \dot{b}_{k,s} + b'_s \hat{X}_s$ and $\hat{\sigma}_{j,s} = \dot{\sigma}_{k,j,s} + \sigma'_{j,s}\hat{X}_s$, where $\dot{b}_{k,s} = \partial_{\alpha_k} b_s$ and $\dot{\sigma}_{k,j,s} = \partial_{\alpha_k}\sigma_{j,s}$. Here again we have $\nabla_{\hat{x}}\hat{b}_s = b'_s$ and $\nabla_{\hat{x}}\hat{\sigma}_{j,s} = \sigma'_{j,s}$, thus the dynamics on $\hat{Y}$ and $\hat{Z}$ are the same as those on Y and Z and we have, for all s, $\hat{Y}_s = Y_s$ and $\hat{Z}_s = Z_s$. Write, for all $k = 1 \dots m$, $U^k_t = \nabla_x \dot{X}_{k,t}$.

In order to compute the Malliavin derivative of $\dot{X}_{k,t}$, we just need to solve (1.1), (3.1), (3.2) and

$$\begin{aligned} U^k_t &= \int_0^t [(\nabla_x \dot{b}_{k,s} + \sum_{l=1}^d \dot{X}_{lk,s}\nabla_x b'_{l,s})Y_s + b'_s U^k_s]ds \\ &+ \sum_{j=1}^d \int_0^t [(\nabla_x \dot{\sigma}_{k,j,s} + \sum_{l=1}^d \dot{X}_{lk,s}\nabla_x \sigma'_{l,j,s})Y_s + \sigma'_{j,s}U^k_s]dW^j_s \end{aligned} \tag{3.14}$$

where $\dot{X}_{lk,s} = \partial_{\alpha_k} X_{l,s}$ is the $(l,k)^{th}$ element of $\dot{X}_s$.

Then, the Malliavin derivative of $\dot{X}_k$ is

$$\mathcal{D}_s(\dot{X}_{k,t}) = \left[(U^k_t - Y_t Z_s U^k_s)Z_s\sigma_s + Y_t Z_s \tilde{\sigma}^k_s\right]\mathbf{1}_{s\leq t} \tag{3.15}$$

with $\tilde{\sigma}^k_s$ the matrix whose j^{th} columns are $\dot{\sigma}_{k,j,s} + \sigma'_{j,s}\dot{X}_{k,s}$.

3.5. Computation of $\frac{1}{t}\sum_i \int_0^t \mathcal{D}_s F_{i,k,t} u_{i,s} ds$. We have

$$\sum_{i=1}^d \mathcal{D}_s F_{i,k,t} u_{i,s} = \mathrm{tr}(u_s \mathcal{D}_s F_{k,t}) = \mathrm{tr}(\sigma_s^{-1} Y_s \mathcal{D}_s F_{k,t})$$

and from (3.6), we deduce

$$\sum_{i=1}^d \int_0^t \mathcal{D}_s F_{i,k,t} u_{i,s} ds = \int_0^t \mathrm{tr}\big(\sigma_s^{-1} Y_s Z_t\big[-\sum_{i=1}^d \mathcal{D}_s(Y_{i,t})(Z_t\dot{X}_{k,t})_i + \mathcal{D}_s(\dot{X}_{k,t})\big]\big)ds.$$

Using equations (3.13) and (3.15) and the property of the trace function $\mathrm{tr}(AB) = \mathrm{tr}(BA)$, it follows that

$$\begin{aligned} \mathrm{tr}\big(\sigma_s^{-1}Y_s Z_t \sum_{i=1}^d \mathcal{D}_s(Y_{i,t})(Z_t\dot{X}_{k,t})_i\big) &= \sum_{i=1}^d (Z_t\dot{X}_{k,t})_i\big[\mathrm{tr}(\sigma_s^{-1}Y_s Z_t V^i_t Z_s \sigma_s) \\ &\quad - \mathrm{tr}(\sigma_s^{-1}V^i_s Z_s\sigma_s) + \mathrm{tr}(\sigma_s^{-1}\sigma^i_s)\big] \\ &= \sum_{i=1}^d (Z_t\dot{X}_{k,t})_i \mathrm{tr}(Z_t V^i_t - Z_s V^i_s + \sigma_s^{-1}\sigma^i_s) \end{aligned}$$

and

$$\begin{aligned} \mathrm{tr}\big(\sigma_s^{-1}Y_s Z_t \mathcal{D}_s(\dot{X}_{k,t})\big) &= \mathrm{tr}(\sigma_s^{-1}Y_s Z_t U^k_t Z_s\sigma_s) - \mathrm{tr}(\sigma_s^{-1}U^k_s Z_s\sigma_s) + \mathrm{tr}(\sigma_s^{-1}\tilde{\sigma}^k_s) \\ &= \mathrm{tr}(Z_t U^k_t - Z_s U^k_s + \sigma_s^{-1}\tilde{\sigma}^k_s). \end{aligned}$$

Thus we have

$$\frac{1}{t}\sum_{i=1}^{d}\int_0^t \mathcal{D}_s F_{i,k,t} u_{i,s} ds = \sum_{i=1}^{d}(Z_t \dot{X}_{k,t})_i \left[-\mathrm{tr}(Z_t V_t^i) + \frac{1}{t}\int_0^t \mathrm{tr}(Z_s V_s^i - \sigma_s^{-1}\sigma_s^i) ds\right]$$
$$+\mathrm{tr}(Z_t U_t^k) + \frac{1}{t}\int_0^t \mathrm{tr}(-Z_s U_s^k + \sigma_s^{-1}\tilde{\sigma}_s^k) ds. \qquad (3.16)$$

3.6. In short .... We solve the following system: for all $t \in [0,T]$,

$$X_t = x + \int_0^t b_s ds + \sum_{j=1}^{d}\int_0^t \sigma_{j,s} dW_s^j$$
$$Y_t = \mathrm{I}_d + \int_0^t b'_s\, Y_s\, ds + \sum_{j=1}^{d}\int_0^t \sigma'_{j,s}\, Y_s\, dW_s^j$$
$$Z_t = \mathrm{I}_d - \int_0^t Z_s(b'_s - \sum_{j=1}^{d}(\sigma'_{j,s})^2)\, ds - \sum_{j=1}^{d}\int_0^t Z_s \sigma'_{j,s}\, dW_s^j$$
$$\dot{X}_t = \int_0^t \left(\dot{b}_s + b'_s\, \dot{X}_s\right) ds + \sum_{j=1}^{d}\int_0^t \left(\dot{\sigma}_{j,s} + \sigma'_{j,s}\, \dot{X}_s\right) dW_s^j$$
$$\text{for all } i = 1\ldots d,\ V_t^i = \int_0^t [(\sum_{l=1}^{d} Y_{li,s}\nabla_x b'_{l,s})Y_s + b'_s V_s^i] ds$$
$$+\sum_{j=1}^{d}\int_0^t [(\sum_{l=1}^{d} Y_{li,s}\nabla_x \sigma'_{l,j,s})Y_s + \sigma'_{j,s} V_s^i] dW_s^j$$
$$\text{for all } k = 1\ldots m,\ U_t^k = \int_0^t [(\nabla_x \dot{b}_{k,s} + \sum_{l=1}^{d} \dot{X}_{lk,s}\nabla_x b'_{l,s})Y_s + b'_s U_s^k] ds$$
$$+\sum_{j=1}^{d}\int_0^t [(\nabla_x \dot{\sigma}_{k,j,s} + \sum_{l=1}^{d} \dot{X}_{lk,s}\nabla_x \sigma'_{l,j,s})Y_s + \sigma'_{j,s} U_s^k] dW_s^j.$$

All these processes are simulated using the Euler scheme with N time steps, which gives a global discretization error of order $\frac{1}{N}$, as proved in the main paper.

Then we compute (3.16) and (3.4) and deduce $H_{k,t}$ from (3.3).

4. Adjoint approach. From Lemma 2.9 and Theorem 2.11 we have

$$\dot{J}(\alpha) = \mathbb{E}\Big[\int_0^T\int_0^t \big[f(X_t) - f(X_s)\big](H_{t,s}^b + H_{t,s}^\sigma) ds dt$$
$$+ \int_0^T \big[g(X_T) - g(X_t)\big](H_{T,t}^b + H_{T,t}^\sigma) dt\Big]$$

with

$$H^b_{t,s} = \Big(\int_s^t dW_u^* \sigma_u^{-1}\, Y_u\Big) \frac{Z_s}{t-s} \dot{b}_s$$
$$H^\sigma_{t,s} = \sum_{i,j=1}^d [\dot{\sigma\sigma^*}]_{ij,s} \left(\frac{2e^j}{t-s} \cdot \Big[Z_s^* \int_{\frac{t+s}{2}}^t (\sigma_u^{-1} Y_u)^* dW_u\Big] \times \frac{e^i}{t-s} \cdot \Big[Z_s^* \int_s^{\frac{t+s}{2}} (\sigma_u^{-1} Y_u)^* dW_u\Big] \right.$$
$$\left. + \frac{e^i}{t-s} \cdot \Big\{ \nabla_x \Big[Z_s^* \int_s^{\frac{t+s}{2}} (\sigma_u^{-1} Y_u)^* dW_u \Big] Z_s e^j \Big\} \right).$$

For an efficient computation of $H^b_{t,s}$ and $H^\sigma_{t,s}$ for all $0 \le s < t \le T$, we memorize along a trajectory discretized at times $\{t_0 = 0 < t_1 < \cdots < t_N = T\}$ the following data:

$$\Big\{ f(X_s), Z_s, \nabla_x Z_s, \dot{b}_s, [\dot{\sigma\sigma^*}]_s,$$
$$I_s := \int_0^s dW_u^* \sigma_u^{-1} Y_u, \quad J_{k,s} := \int_0^s dW_u^* \partial_{x_k}[\sigma_u^{-1} Y_u] \Big\}_{\substack{s \in \{t_1, t_2, \dots, t_N\}, \\ k \in \{1, \dots, d\}}}$$

Then, after some computations, we derive $H^b_{t,s}$ and $H^\sigma_{t,s}$ for all discrete times:

$$H^b_{t,s} = \frac{I_t - I_s}{t-s} Z_s \dot{b}_s$$
$$H^\sigma_{t,s} = \sum_{i,j=1}^d [\dot{\sigma\sigma^*}]_{ij,s} \left\{ \frac{2}{(t-s)^2} \big(I_t - I_{\frac{t+s}{2}}\big) Z_{j,s} \big(I_{\frac{t+s}{2}} - I_s\big) Z_{i,s} \right.$$
$$\left. + \frac{1}{t-s} \left(\sum_k Z_{kj,s} \Big[J_{k,\frac{t+s}{2}} - J_{k,s} - \big(I_{\frac{t+s}{2}} - I_s\big) Z_s \partial_{x_k} Y_s \Big] \right) Z_{i,s} \right\}.$$

As mentioned in [GM02], if there is no instantaneous cost (i.e. $f = 0$) then it is not necessary to compute $H^b_{t,s}$ and $H^\sigma_{t,s}$ for all t and s: only $H^b_{T,t}$ and $H^\sigma_{T,t}$, for all t, are required and may be computed directly.

5. Martingale approach. From Theorem 2.12 we have

$$\dot{J}(\alpha) = \mathbb{E}\Big[\int_0^T \Big(f(X_t) H_t + \int_0^t \big[f(X_t) - f(X_s)\big] H_{t,s} ds \Big) dt$$
$$+ f(X_T) H_T + \int_0^T \big[f(X_T) - f(X_t)\big] H_{T,t} dt \Big]$$

with

$$H_t = \frac{1}{t} \int_0^t dW_s^* \sigma_s^{-1} \dot{X}_s$$
$$H_{t,s} = \frac{1}{(t-s)^2} \int_s^t dW_u^* \sigma_u^{-1} (\dot{X}_u - Y_u Z_s \dot{X}_s).$$

Here, we memorize the following data along the trajectory:

$$\Big\{ f(X_s), Z_s \dot{X}_s,\ I_s := \int_0^s dW_u^* \sigma_u^{-1} Y_u,\ K_s := \int_0^s dW_u^* \sigma_u^{-1} \dot{X}_u \Big\}_{s \in \{t_1, t_2, \dots, t_N\}}$$

from which we compute H_t and $H_{t,s}$ for all t, s.

REFERENCES

[GM02] E. Gobet and R. Munos. Sensitivity analysis using it-malliavin calculus and martingales. application to stochastic optimal control. *Technical Report 499, CMAP, Ecole Polytechnique*, 2002.